ФІЗІКА

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Intrinsic Symmetries for the Generalized Majorana Fields in Electromagnetic and Gravitational Fields

We start with the generalized multicomponent matrix equation $(\Gamma_\mu \partial_\mu + m)\psi = 0$ *, and introduce the concept of the intrinsic symmetry. These symmetries should preserve the form of the basic equation, this leads to the commutation relation* $[Q, \Gamma_\mu]_- = 0$*. The relevant Lagrangian should be invariant under the intrinsic symmetries, which gives the restriction* $Q^+ \eta Q = \eta$ *for the matrix of the bilinear form* η*. We impose the additional requirement: such symmetries should preserve the Majorana nature of the field. This means that if the function* Ψ_A *is real (imaginary) part of the complete wave function, then while transforming the function* $\Psi'_A = Q_{AB}\Psi_B$ *remains real (imaginary). The main accent is given to multicomponent Majorana fields, which can be related to one, two, three and four Dirac fields; these are closely related to the Dirac – Kähler field.*

First we examine the case of classical fields, and find the generators for the corresponding intrinsic symmetries. Finally, we take into account the presence of external electromagnetic fields, also we extend this approach to Riemannian space-time geometry. Additionally, we consider in detail the relations between performed symmetry analysis for the system 4 Dirac fields and the Dirac – Kähler field. In the end, t we take into account the presence of external electromagnetic and gravitational fields.

Key words: *the systems of Dirac fields, intrinsic symmetry, Majorana fields, Lagrangian formalism, external electromagnetic fields, the Riemannian space-time geometry, Dirac – Kähler particle.*

ВНУТРЕННИЕ СИММЕТРИИ ОБОБЩЕННЫХ МАЙОРАНОВСКИХ ПОЛЕЙ В ЭЛЕКТРОМАГНИТНОМ И ГРАВИТАЦИОННОМ ПОЛЯХ

Рассмотрим обобщенное многокомпонентное матричное уравнение $(\Gamma_\mu \partial_\mu + m)\psi = 0$ *и введем понятие внутренних симметрий. Эти симметрии должны сохранять форму основного уравнения, что приводит к коммутационному соотношению* $[Q, \Gamma_\mu]_- = 0$*. Соответствующий лагранжиан должен быть инвариантен относительно преобразований симметрии, что дает ограничение* $Q^+ \eta Q = \eta$ *на матрицу билинейной формы* η*. Мы накладываем дополнительное требование, чтобы такие симметрии сохраняли майорановский характер поля. Это означает, что если функция* Ψ_A *является вещественной (мнимой) частью полной волновой функции, то после преобразований* $\Psi'_A = Q_{AB}\Psi_B$ *она должна оставаться вещественной (мнимой). Основной акцент делается на многокомпонентных майорановских полях, которые могут быть связаны с одним, двумя, тремя и четырьмя дираковскими полями; последние тесно связаны с полем Дирака – Келера.*

Сначала мы рассматриваем случай классических полей и находим генераторы для соответствующих внутренних симметрий. В конце мы учитываем наличие внешних электромагнитных полей, а также распространяем этот подход на риманову геометрию пространства – времени. Кроме того, мы подробно рассматриваем связи между проведенным анализом симметрий и полем Дирака – Келера.

Ключевые слова: *система дираковских полей, внутренняя симметрия, Майорановские поля, Лагранжев формализм, внешние электромагнитные поля, Риманова геометрия, частица Дирака – Кэлера.*

1. Introduction

The theory of relativistic wave equations gives the base for describing the elementary particles and their interactions. It started with investigations of P. A. M. Dirac [1], W. Pauli [2], and M. Fierz [3]. These studies were proceeded by H. J. Bhabha [4; 5] and Harish-Chandra [6; 7]. They proposed for description of any particle to apply the systems of first order differential equations in matrix form

$$(\Gamma_\mu \partial_\mu + m)\Psi = 0, \tag{1}$$

where Ψ stands for the multicomponent wave functions, Γ_μ designates square matrices, m is the mass parameter. Invariant properties of these matrices can be associated with physical characteristics of the particles.

In this field, very important investigations were performed by I. M. Gelfand and A. M. Yaglom [8] in which the general method for constructing the wave equations in matrix form (1) was developed, for particles with any sets of spin and mass states. They derived general restrictions on the matrices Γ_μ, and several examples of application of this general theory.

Substantial contribution in this theory was made by F. I. Fedorov [9–17]. In particular, he elaborated the method of projective operators [9], which permits to perform complicated calculations without the use of explicit form of the matrices Γ_μ. Important contribution in study of the algebras of the matrices Γ_μ and development of the methods of calculation was done by L. A. Shelepin [18].

Substantial contribution in this field was given by V. I. Fuschich and A. G. Nikitin [19]. They proved the existence of intrinsic symmetries for many relativistic wave equations.

There exists one special way for describing the intrinsic degrees of freedom and additional characteristics of the particles within the general theory of relativistic wave equations. It is based on the use of extended (repeated) sets of representations of the Lorentz group [20] when constructing generalized wave equations for particles.

The known example of such a wave equation is the Dirac – Kähler system referring to the particle with two spin states ($s = 0,1$) and degeneration in intrinsic parities [21–24]. Firstly, the Dirac – Kähler equation was formulated in tensor form by C. G. Darwin [25]. He proposed for describing the electron in external magnetic field to apply the complicated tensor system of equations. In this way, he derived the energy spectrum in presence of external Coulomb field, which coincides with that in the Dirac theory. Later on, this Darwin equation was rediscovered by many researchers. The best known is the paper by E. Kähler [26], where the formalism of differential forms was used. In the papers of V. I. Strazhev with coauthors [27; 28], this system was studied in detail within the conventional theory relativistic wave equations. In the literature, in different mathematical approaches [29–31], they discuss intrinsic symmetries of the Dirac equation, which differ from space-time symmetries.

2. Basic Definitions

Let us consider an arbitrary relativistic wave equation for a particle with nonzero mass in the standard form

$$(\Gamma_\mu \partial_\mu + m)\psi = 0, \tag{2}$$

where Ψ is multicomponent wave function, Γ_μ stands for squared matrices, m is the mass parameter. For this equation, there exists the Lagrangian

$$L = -\psi^{+}\eta(\Gamma_{\mu}\partial_{\mu} + m)\psi. \quad (3)$$

Under the intrinsic symmetry we mean the linear transformation

$$\Psi'_{A} = Q_{AB}\Psi_{B}, \quad (4)$$

which obeys a number of conditions.

First, they should preserve the form of the basic equation (2). Acting by the matrix Q on eq. (2), if the commutator is valid

$$[Q, \Gamma_{\mu}]_{-} = 0, \quad (5)$$

we obtain

$$(\Gamma_{\mu}\partial_{\mu} + m)Q\psi = 0 \quad \Rightarrow \quad (\Gamma_{\mu}\partial_{\mu} + m)\psi' = 0.$$

Restriction (5) ensues invariance of eq. (2) under the transformation (4). Second, the Lagrangian (3) should be invariant under the transformation (4) as well:

$$L' = -(\psi^{+})'\eta(\Gamma_{\mu}\partial_{\mu} + m)\psi' = -(Q\psi)^{+}\eta(\Gamma_{\mu}\partial_{\mu} + m)Q\psi$$
$$= -\psi^{+}Q^{+}\eta(\Gamma_{\mu}\partial_{\mu} + m)Q\psi = L,$$

whence it follows $\psi^{+}Q^{+}\eta(\Gamma_{\mu}\partial_{\mu} + m)Q\psi = \psi^{+}\eta(\Gamma_{\mu}\partial_{\mu} + m)\psi,$ or differently

$$Q^{+}\eta(\Gamma_{\mu}\partial_{\mu} + m)Q = \eta(\Gamma_{\mu}\partial_{\mu} + m).$$

Thus, the intrinsic symmetry transformations are to obey the constraints

$$Q^{+}\eta\Gamma_{\mu}Q = \eta\Gamma_{\mu}, \quad Q^{+}\eta Q = \eta. \quad (6)$$

Bearing in mind (5), we can conclude that two relations in (6) coincide

$$Q^{+}\eta\Gamma_{\mu}Q = Q^{+}\eta Q\Gamma_{\mu} = \eta\Gamma_{\mu} \quad \Rightarrow \quad Q^{+}\eta Q = \eta. \quad (7)$$

We will impose an additional requirement on symmetry transformations. Such transformations should preserve the Majorana nature of the fields. This means that if the function Ψ_{A} is real (imaginary) part of the complete wave function, then after symmetry transformation the function $\Psi'_{A} = Q_{AB}\Psi_{B}$ remains real (imaginary). In the following, this requirement is called the Majorana condition.

Now let us specify the massless case

$$\Gamma_{\mu}\partial_{\mu}\psi = 0. \quad (8)$$

Requirement of invariance of this equation leads to two alternative restrictions

$$\Gamma_{\mu}\partial_{\mu}Q_{1}\psi = 0 \Rightarrow [Q_{1}, \Gamma_{\mu}]_{-} = 0; \quad (9)$$
$$-\Gamma_{\mu}\partial_{\mu}Q_{2}\psi = 0 \Rightarrow [Q_{2}, \Gamma_{\mu}]_{+} = 0. \quad (10)$$

Additional requirement of the Lagrangian invariance also leads to two possibilities. The first is

$$L' = L, \quad \left[Q_{1}, \Gamma_{\mu}\right]_{-} = 0; \quad (11)$$

it coincides with the yet known constraint (7), which appears when considering the massive case. The second possibility is

$$L' = -L, \quad \left[Q_2, \Gamma_\mu\right]_+ = 0; \tag{12}$$

whence we obtain

$$-\psi^+ Q_2^+ \eta \Gamma_\mu \partial_\mu Q_2 \psi = +\psi^+ \eta \Gamma_\mu \partial_\mu \psi,$$

or $Q_2^+ \eta \Gamma_\mu Q_2 = -\eta \Gamma_\mu$. Bearing in mind the relation $\left[Q_2, \Gamma_\mu\right]_+ = 0$, we conclude that the last relation is equivalent to the known restriction (7). Thus, the Lagrangian invariance with respect to intrinsic symmetry both for massive and massless cases assumes one the same constraint (7):

$$Q^+ \eta Q = \eta.$$

For infinitesimal one-parametric symmetry transformations

$$Q = 1 + \omega J \tag{13}$$

relation (7) takes on the form

$$(\omega J)^+ \eta = -\eta\, \omega J. \tag{14}$$

3. One Dirac Equation in Electromagnetic Field

Let us take into account the presence of external electromagnetic field

$$(\gamma_\mu \partial_\mu - ie\gamma_\mu A_\mu + m)\psi = 0, \tag{15}$$

where e stands for electric charge, and $A_\mu = (A_1, A_2, A_3, A_4)$ is an electromagnetic 4-potential, $(A_4 = i\varphi)$. Let us separate in the wave function real and imaginary parts:

$$\gamma_i^* = \gamma_i, \gamma_4^* = -\gamma_4, \partial_i^* = \partial_i, \partial_4^* = -\partial_4, A_i^* = A_i, A_4^* = -A_4$$

so we have two equations

$$\begin{gathered}(\gamma_\mu \partial_\mu - ie\gamma_\mu A_\mu + m)\psi = 0,\\ (\gamma_\mu \partial_\mu + ie\gamma_\mu A_\mu + m)\psi^* = 0.\end{gathered} \tag{16}$$

Summing and subtracting them, for 8-component function Ψ (19) we obtain an equation in the standard matrix form

$$(\Gamma_\mu \partial_\mu - ieG_\mu A_\mu + m)\Psi = 0, \tag{17}$$

where $\Gamma_\mu = I_2 \otimes \gamma_\mu$, and additional matrices G_μ are

$$G_\mu = \sigma_1 \otimes \gamma_\mu. \tag{18}$$

Note that the last equation contains two sets of matrices, Γ_μ and G_μ.

Now we will search for restrictions on intrinsic symmetries Q, in presence of electromagnetic fields they may differ from previously established for a free particle. Such transformations $\Psi' = Q\Psi$ should preserve the form of the main equation (17):

$$(\Gamma_\mu \partial_\mu - ieG_\mu A_\mu + m)Q\Psi = 0 \quad \Rightarrow \quad (\Gamma_\mu \partial_\mu - ieG_\mu A_\mu + m)\Psi' = 0.$$

So, there arise two constraints

$$[Q, \Gamma_\mu]_- = 0, \tag{19}$$

$$[Q, G_\mu]_- = 0. \tag{20}$$

The Lagrangian for the case under consideration has the structure

$$L = -\Psi^{+}\eta(\Gamma_{\mu}\partial_{\mu} - ieG_{\mu}A_{\mu} + m)\Psi. \tag{21}$$

Let this structure be invariant with respect to transformation (4):

$$L' = -(\Psi')^{+}\eta(\Gamma_{\mu}\partial_{\mu} - ieG_{\mu}A_{\mu} + m)Q\Psi' = (Q\Psi)^{+}\eta(\Gamma_{\mu}\partial_{\mu} - ieG_{\mu}A_{\mu} + m)Q\Psi =$$
$$= -\Psi^{+}Q^{+}\eta(\Gamma_{\mu}\partial_{\mu} - ieG_{\mu}A_{\mu} + m)Q\Psi = -\Psi^{+}\eta(\Gamma_{\mu}\partial_{\mu} - ieG_{\mu}A_{\mu} + m)\Psi = L,$$

so we derive

$$\Psi^{+}Q^{+}\eta(\Gamma_{\mu}\partial_{\mu} - ieG_{\mu}A_{\mu} + m)Q\Psi = \Psi^{+}\eta(\Gamma_{\mu}\partial_{\mu} - ieG_{\mu}A_{\mu} + m)\Psi,$$

whence (bearing in mind (19) and (20)) it follows

$$Q^{+}\eta Q = \eta. \tag{22}$$

The last condition coincides with the previously established in absence of external fields; its consequences were studied in the above.

Thus, the presence of external electromagnetic fields leads to one additional constraint (20). For infinitesimal transformations $Q = 1 + \omega J$ it takes the form

$$[J, G_{\mu}]_{-} = 0, \tag{23}$$

where J stands for any generator related to intrinsic symmetries.

Now we should verify which generators from established in the above (in the case of one Dirac field) satisfy the constraint (23). In Majorana basis ψ^{r}, ψ^{i} (17), there exists only one generator which satisfy restrictions (19) and (22):

$$J_1 = \sigma_1 \otimes I_4. \tag{24}$$

By direct calculation, we can verify that p J_1 obeys the above additional constrain (23). In other words, the intrinsic symmetry in presence of external field is the same as in the case of a free particle.

4. The Case of n Dirac Fields

Let us consider n Dirac fields in presence of electromagnetic fields

$$(\gamma_{\mu}\partial_{\mu} - ie\gamma_{\mu}A_{\mu} + m)\psi_k = 0, \quad k = 1,...,n. \tag{25}$$

The conjugate equation is

$$(\gamma_{\mu}\partial_{\mu} + ie\gamma_{\mu}A_{\mu} + m)\psi_k^{*} = 0; \tag{26}$$

further we obtain

$$\begin{aligned} &\gamma_{\mu}\partial_{\mu}(\psi_k + \psi_k^{*}) - ie\gamma_{\mu}A_{\mu}(\psi_k - \psi_k^{*}) + m(\psi_k + \psi_k^{*}) = 0, \\ &\gamma_{\mu}\partial_{\mu}(\psi_k - \psi_k^{*}) - ie\gamma_{\mu}A_{\mu}(\psi_k + \psi_k^{*}) + m(\psi_k - \psi_k^{*}) = 0. \end{aligned} \tag{27}$$

Thus, we have $(n+n)$ new equations

$$\begin{aligned}
&\gamma_\mu \partial_\mu \psi_1^r - ie\gamma_\mu A_\mu \psi_1^i + m\psi_1^r = 0,\\
&\dots\\
&\gamma_\mu \partial_\mu \psi_n^r - ie\gamma_\mu A_\mu \psi_n^i + m\psi_n^r = 0;\\
&\gamma_\mu \partial_\mu \psi_1^i - ie\gamma_\mu A_\mu \psi_1^r + m\psi_1^i = 0,\\
&\dots\\
&\gamma_\mu \partial_\mu \psi_n^i - ie\gamma_\mu A_\mu \psi_n^r + m\psi_n^i = 0.
\end{aligned} \tag{28}$$

With the use of the $2n$-component wave function Ψ with the structure

$$\Psi = (\psi_1^r, \psi_2^r, \dots, \psi_n^r; \psi_1^i, \psi_2^i, \dots, \psi_n^i) = (\psi^r, \psi^i) \text{ the column,} \tag{29}$$

we can present the above system (28) as follows

$$\begin{vmatrix} \gamma_\mu & \\ & \gamma_\mu \end{vmatrix} \partial_\mu \begin{vmatrix} \psi^r \\ \psi^i \end{vmatrix} - ie \begin{vmatrix} & \gamma_\mu \\ \gamma_\mu & \end{vmatrix} A_\mu + m \begin{vmatrix} \psi^r \\ \psi^i \end{vmatrix} = 0, \tag{30}$$

or in the matrix form

$$(\Gamma_\mu \partial_\mu - ieG_\mu A_\mu + m)\Psi = 0, \tag{31}$$

where

$$\Gamma_\mu = I_2 \otimes (I_n \otimes \gamma_\mu), \quad G_\mu = \sigma_1 \otimes (I_n \otimes \gamma_\mu). \tag{32}$$

Now, we should find restrictions on the intrinsic symmetries Q imposed by the requirements (19) and (20).

From the constraint $[Q, \Gamma_\mu]_- = 0$ (see (19)), taking into account the structure of Γ_μ (32), we get the possible structure of the transformation Q:

$$Q = A \otimes (B \otimes I_4), \tag{33}$$

where A and B are complex matrices with dimensions 2×2 and $n \times n$ respectively. Indeed, we readily prove the identity (it is valid for any A and B)

$$\begin{gathered}
[Q, \Gamma_\mu]_- = [A \otimes B \otimes I_4, I_2 \otimes I_n \otimes \gamma_\mu]_- = (A \otimes B \otimes I_4)(I_2 \otimes I_n \otimes \gamma_\mu) - \\
-(I_2 \otimes I_n \otimes \gamma_\mu)(A \otimes B \otimes I_4) = A \otimes (B \otimes I_4)(I_n \otimes \gamma_\mu) - \\
-A \otimes (I_n \otimes \gamma_\mu)(B \otimes I_4) = A \otimes (B \otimes \gamma_\mu) - A \otimes (B \otimes \gamma_\mu) = 0.
\end{gathered}$$

Substituting (33) into (20):

$$\begin{gathered}
[Q, G_\mu]_- = [A \otimes B \otimes I_4, \sigma_1 \otimes I_n \otimes \gamma_\mu]_- = (A \otimes B \otimes I_4)(\sigma_1 \otimes I_n \otimes \gamma_\mu) - \\
-(\sigma_1 \otimes I_n \otimes \gamma_\mu)(A \otimes B \otimes I_4) = \sigma_1 A \otimes (I_n \otimes \gamma_\mu)(B \otimes I_4) - A\sigma_1 (B \otimes I_4)(I_n \otimes \gamma_\mu) = \\
= \sigma_1 A \otimes (B \otimes \gamma_\mu) - A\sigma_1 \otimes (B \otimes \gamma_\mu) = (\sigma_1 A - A\sigma_1) \otimes (B \otimes \gamma_\mu) = 0,
\end{gathered}$$

we get the restriction on the matrix A:

$$[\sigma_1, A]_- = 0; \tag{34}$$

for the matrix B no restrictions arise.

Thus the intrinsic symmetry Q should have the structure (33) with additional constraint (34). It is evident that the corresponding generators should obey the structure similar to (33):

$$J = a \otimes (b \otimes I_4), \tag{35}$$

where a and b are complex matrices of dimensions 2×2 and $n\times n$ respectively besides, they should satisfy the constraint

$$[\sigma_1, a]_- = 0. \tag{36}$$

For the case of one Dirac equation (15), there exists only one generator (24); $a = \sigma_1$ and eq. $[\sigma_1, a]_- = 0$ is satisfied, and $b = 1$.

5. Two Dirac Fields in Electromagnetic Fields

In the case of two Dirac fields, the matrices Γ_μ and G_μ have the form

$$\Gamma_\mu = I_2 \otimes (I_2 \otimes \gamma_\mu), \quad G_\mu = \sigma_1 \otimes (I_2 \otimes \gamma_\mu). \tag{37}$$

The intrinsic symmetry generators for free fields were found in the above

$$\begin{aligned} J_1 = \gamma_1 \otimes I_4, \quad J_3 = \gamma_3 \otimes I_4, \quad J_{11} = i\gamma_3\gamma_1 \otimes I_4, \\ J_7 = i\gamma_2\gamma_5 \otimes I_4, \quad J_9 = i\gamma_4\gamma_5 \otimes I_4, \quad J_{14} = i\gamma_2\gamma_4 \otimes I_4. \end{aligned} \tag{38}$$

It is convenient to present them differently. To this end, we take into account eq. (16), $\gamma_5 = -\sigma_2 \otimes I_2$ and the multiplication rule $\sigma_i\sigma_j = i\varepsilon_{ijk}\sigma_k + \delta_{ij}$. Then we obtain

$$\gamma_3\gamma_1 = (\sigma_1 \otimes \sigma_3)(\sigma_1 \otimes \sigma_1) = \sigma_1\sigma_1 \otimes \sigma_3\sigma_1 = iI_2 \otimes \varepsilon_{312}\sigma_2 = iI_2 \otimes \sigma_2,$$

similarly

$$\gamma_2\gamma_5 = i\sigma_1 \otimes I_2, \quad \gamma_2\gamma_4 = i\sigma_2 \otimes \sigma_2, \quad \gamma_4\gamma_5 = -i\sigma_3 \otimes \sigma_2.$$

With the last relations in mind, the generators (48) are presented as follows

$$\begin{aligned} J_1 = \sigma_1 \otimes (\sigma_1 \otimes I_4), J_3 = \sigma_1 \otimes (\sigma_3 \otimes I_4), J_{11} = -I_2 \otimes (\sigma_2 \otimes I_4), \\ J_7 = -\sigma_1 \otimes (I_2 \otimes I_4), J_9 = \sigma_3 \otimes (\sigma_2 \otimes I_4), J_{14} = -\sigma_2 (\otimes \sigma_2 \otimes I_4). \end{aligned} \tag{39}$$

All six generators have the structure $J = a \otimes (b \otimes I_4)$, where

$$a, b \in \{\sigma_1, \sigma_2, \sigma_3, I_2\}. \tag{40}$$

Evidently, the additional condition $[a, \sigma_1]_= 0$ is satisfied only for generators of the type $a \in \{\sigma_1, I_2\}$; so remain only 4 generators

$$J_1, J_3, J_{11}, J_7, \tag{41}$$

note that J_1, J_3, J_{11} obey the Lie algebra $su(2)$. The generator J_7 commutes with J_1, J_3, J_{11}.

Thus, we have 4-parametric group with the structure $SU(2)\otimes U(1)$. Taking in mind the explicit form of the J_7

$$J_7 = -\sigma_1 \otimes (I_2 \otimes I_4) = -\sigma_1 \otimes I_8,$$

we readily find expression for corresponding finite transformation Q_7:

$$Q_7 = e^{i\sigma_1\Omega} \otimes I_8 = (\cos\Omega + i\sin\Omega\,\sigma_1) \otimes I_8. \qquad (42)$$

6. Three Dirac Equations in Electromagnetic Fields

For three Dirac equations fields (64), (65) in presence of external electromagnetic field, the matrices Γ_μ and G_μ in (31) take the form

$$\Gamma_\mu = I_2 \otimes I_3 \otimes \gamma_\mu, \quad G_\mu = \sigma_1 \otimes I_3 \otimes \gamma_\mu. \qquad (43)$$

The intrinsic symmetry generators in absence of electromagnetic fields were found in the above, they are

$$\begin{aligned} &J_1 = (\sigma_1 \otimes I_3) \otimes I_4, \quad J_9 = (I_2 \otimes \alpha_6) \otimes I_4, \quad J_{10} = (I_2 \otimes \alpha_7) \otimes I_4, \\ &J_{11} = (I_2 \otimes \alpha_8) \otimes I_4, \quad J_{12} = (\sigma_1 \otimes \alpha_1) \otimes I_4, \quad J_{13} = (\sigma_1 \otimes \alpha_2) \otimes I_4, \\ &J_{14} = (\sigma_1 \otimes \alpha_3) \otimes I_4, \quad J_{15} = (\sigma_1 \otimes \alpha_4) \otimes I_4, \quad J_{16} = (\sigma_1 \otimes \alpha_5) \otimes I_4, \\ &J_{25} = (\sigma_2 \otimes \alpha_6) \otimes I_4, \quad J_{26} = (\sigma_2 \otimes \alpha_7) \otimes I_4, \quad J_{27} = (\sigma_2 \otimes \alpha_8) \otimes I_4, \\ &J_{33} = (\sigma_3 \otimes \alpha_6) \otimes I_4, \quad J_{34} = (\sigma_3 \otimes \alpha_7) \otimes I_4, \quad J_{35} = (\sigma_3 \otimes \alpha_8) \otimes I_4. \end{aligned}$$

All these generators have the structure $J = a \otimes (b \otimes I_4)$, where $a \in \{\sigma_1, \sigma_2, \sigma_3, I_2\}$, and 3-dimensional matrices b are Gell-Mann matrices, $b \in \{\alpha_1, ..., \alpha_8, I_3\}$.

Additional condition (34) is satisfied only for generators of the type $a \in \{\sigma_1, I_2\}$:

$$J_1, J_9, J_{10}, J_{11}, J_{12}, J_{13}, J_{14}, J_{15}, J_{16}; \qquad (44)$$

they make up a 9-parametric group, in which we can separate two noncommuting subgroups with the structure SU(2): J_9, J_{13}, J_{14} and J_{10}, J_{12}, J_{15}. On the matrices b no restrictions arise.

7. Four Dirac Fields

For the case of 4 Dirac fields, the matrices Γ_μ and G_μ have the structure

$$\Gamma_\mu = I_2 \otimes (I_4 \otimes \gamma_\mu), \quad G_\mu = \sigma_1 \otimes (I_4 \otimes \gamma_\mu). \qquad (45)$$

The generators of the intrinsic symmetry in absence of external fields were found in the above. Let us transform these generators to the structure $J = a \otimes (b \otimes I_4)$, and find the relevant matrices a:

$$\begin{aligned} &J_1 \to a = \sigma_1, \quad J_3 \to a = \sigma_3, \quad J_7 \to a = \sigma_1, \quad J_9 \to a = \sigma_3, \\ &J_{11} \to a = I_2, \quad J_{14} \to a = \sigma_2, \quad J_{16} \to a = \sigma_1, \quad J_{18} \to a = \sigma_1, \\ &J_{20} \to a = \sigma_3, \quad J_{22} \to a = \sigma_1, \quad J_{24} \to a = \sigma_1, \quad J_{26} \to a = \sigma_1, \\ &J_{29} \to a = \sigma_1, \quad J_{32} \to a = \sigma_3, \quad J_{34} \to a = \sigma_1, \quad J_{36} \to a = \sigma_1, \\ &J_{38} \to a = \sigma_3, \quad J_{40} \to a = \sigma_3, \quad J_{42} \to a = \sigma_3, \quad J_{44} \to a = \sigma_2, \\ &J_{46} \to a = I_2, \quad J_{48} \to a = I_2, \quad J_{50} \to a = \sigma_2, \quad J_{53} \to a = I_2, \\ &J_{55} \to a = \sigma_2, \quad J_{57} \to a = \sigma_2, \quad J_{59} \to a = I_2, \quad J_{62} \to a = I_2. \end{aligned}$$

Evidently, the additional condition $[a, \sigma_1]_- = 0$ will be satisfied only for generators of the type $a \in \{\sigma_1, I_2\}$; they are the following 15 generators

$$J_1, J_7, J_{11}, J_{16}, J_{18}, J_{22}, J_{24}, J_{26}, J_{34}, J_{36}, J_{46}, J_{48}, J_{53}, J_{59}, J_{62}. \qquad (46)$$

On the matrices b no additional restrictions in presence of external electromagnetic fields arise.

The study of the explicit structure of generators (46) permits to get the following conclusions:

1. Among these generators we can see 16 triples, each of them obeys the commutative relations of the group $SU(2)$

$$\begin{aligned}&(J_1,J_{22},J_{46}),(J_1,J_{24},J_{48}),(J_1,J_{26},J_{53}),(J_{11},J_{16},J_{22}),\\&(J_{11},J_{18},J_{24}),(J_{11},J_{53},J_{59}),(J_{16},J_{18},J_{62}),(J_{16},J_{36},J_{59}),\\&(J_{18},J_{34},J_{59}),(J_{22},J_{24},J_{62}),(J_{22},J_{36},J_{53}),(J_{24},J_{34},J_{53}),\\&(J_{26},J_{34},J_{48}),(J_{26},J_{36},J_{46}),(J_{34},J_{36},J_{62}),(J_{46},J_{48},J_{62}).\end{aligned} \quad (47)$$

2. Among these triples (47), there exist 6 pairs, commuting with each other:

$$\begin{aligned}&(J_1,J_{22},J_{46}),(J_{18},J_{34},J_{59});\\&(J_1,J_{24},J_{48}),(J_{16},J_{36},J_{59});\\&(J_1,J_{26},J_{53}),(J_{16},J_{18},J_{62});\\&(J_{11},J_{16},J_{22}),(J_{26},J_{34},J_{48});\\&(J_{11},J_{18},J_{24}),(J_{26},J_{36},J_{46});\\&(J_{11},J_{53},J_{59}),(J_{46},J_{48},J_{62}).\end{aligned} \quad (48)$$

3. The triples

$$(J_{22},J_{24},J_{62}),\quad (J_{22},J_{36},J_{53}),\quad (J_{24},J_{34},J_{53}),\quad (J_{34},J_{36},J_{62})$$

do not have such pairs.

8. Dirac Equation in Riemannian Space-Time

Initial Dirac equation in Minkowski space (x_1,x_2,x_3,ict)

$$(\gamma_\mu\partial_\mu+m)\psi=0 \quad (49)$$

in presence of a curved space-time background has the form

$$[i\gamma'^\alpha(x)(\partial_\alpha+B_\alpha(x))-m]\psi(x)=0, \quad (50)$$

where

$$\gamma'^\alpha(x)=\gamma'^a e^\alpha_{(a)},\quad e^\alpha_{(a)}\text{isatetrad},$$

$$B_\alpha(x)=\frac{1}{2}\sigma^{ab}e^\beta_{(a)}\nabla_\alpha(e_{(b)\beta}),\quad \sigma^{ab}=\frac{1}{4}(\gamma'^a\gamma'^b-\gamma'^b\gamma'^a),$$

or

$$B_\alpha(x)=\frac{1}{4}\gamma'^\beta(x)\nabla_\alpha\gamma'_\beta(x);$$

∇_α stands for the covariant derivative, the metrical signature is $g_{ab}=(1,-1,-1,-1)$. The Dirac matrices in metrical representation γ' relate to the matrices (16) as follows

$$\gamma'^0=\gamma_4,\quad \gamma'^1=i\gamma_1,\quad \gamma'^2=i\gamma_2,\quad \gamma'^3=i\gamma_3. \quad (51)$$

In metrical representation, the Dirac matrices in Majorana basis are determined by the formulas

$$\gamma'^1 = \begin{vmatrix} & & & i \\ & & i & \\ & i & & \\ i & & & \end{vmatrix}, \gamma'^2 = \begin{vmatrix} i & & & \\ & i & & \\ & & -i & \\ & & & -i \end{vmatrix},$$
$$\gamma'^3 = \begin{vmatrix} & & i & \\ & & & -i \\ i & & & \\ & -i & & \end{vmatrix}, \gamma'^0 = \begin{vmatrix} & & & -i \\ & & i & \\ & -i & & \\ i & & & \end{vmatrix}, \tag{52}$$

they are imaginary, under the Hermitian conjugation they behave as follows

$$(\gamma'^0)^+ = \gamma'^0, \quad (\gamma'^1)^+ = -\gamma'^1, \quad (\gamma'^2)^+ = -\gamma'^2, \quad (\gamma'^3)^+ = -\gamma'^3. \tag{53}$$

In accordance with the above definitions, the wave operator in the Dirac equation (50) in Majorana basis is real; this means that there exist two types of Majorana fermions in Riemannian space as well as in Minkowski space.

The Lagrangian for Dirac field in Riemannian space is determined as follows

$$L = -\psi^+\eta(i\gamma'^\mu(x)D_\mu - m)\psi, \eta = \gamma'^0, \eta^+ = \eta, D_\mu = \partial_\mu + B_\mu(x); \tag{54}$$

note the commutation rule

$$D_\mu\gamma'^\mu(x) = \gamma'^\mu(x)D_\mu. \tag{55}$$

It is evident that in presence of gravitational fields (in presence of any curved space-time background), the basic relations which determine the properties of the intrinsic symmetries, preserve their form

$$[Q, \gamma'^\mu(x)]_- = 0, \quad Q^+\eta Q = \eta. \tag{56}$$

Therefore, all above established properties of intrinsic symmetries and their classification are valid in Riemannian space-time models as well, when considering 1, 2, 3, and 4 Dirac fields, both for massive and massless particles.

9. System of 4 Dirac fields, relation to the Dirac – Kähler particle, the metric g_{ab}=diag(1, -1, -1 -1)

In this section, we will use the metric tensor with the signature $(1,-1,-1,-1)$; then the Dirac equation in Majorana basis reads

$$(i\gamma^\mu\partial_\mu - m)\psi = 0, \tag{57}$$

now the Dirac matrices are imaginary

$$\gamma^0 = \sigma_1 \otimes \sigma_2, \quad \gamma^1 = i\sigma_1 \otimes \sigma_1, \quad \gamma^2 = i\sigma_3 \otimes I_2, \quad \gamma^3 = i\sigma_1 \otimes \sigma_3. \tag{58}$$

Let us turn to four Dirac-like equations

$$(i\gamma^\mu\partial_\mu - m)\psi_k = 0, \quad k = 1, 2, 3, 4; \tag{59}$$

it is convenient to present these 4 bispinor functions as the columns of the 4×4 matrix

$$(\psi_k) = (\psi_1, \psi_2, \psi_3, \psi_4) = \begin{vmatrix} \psi_{11} & \psi_{21} & \psi_{31} & \psi_{41} \\ \psi_{12} & \psi_{22} & \psi_{32} & \psi_{42} \\ \psi_{13} & \psi_{23} & \psi_{33} & \psi_{43} \\ \psi_{14} & \psi_{24} & \psi_{34} & \psi_{44} \end{vmatrix}. \tag{60}$$

Consider conjugate equations

$$(i\gamma^\mu \partial_\mu - m)\psi_k^* = 0. \tag{61}$$

After summing and subtracting functions in relevant pairs

$$\psi_k^r = \frac{1}{\sqrt{2}}(\psi_k + \psi_k^*), \; \psi_k^i = \frac{1}{\sqrt{2}}(\psi_k - \psi_k^*), \tag{62}$$

we get the system in the standard form

$$(i\Gamma^\mu \partial_\mu - m)\Psi = 0, \tag{63}$$

where Ψ is a 32-component wave function

$$\Psi = (\psi_1^r, \psi_2^r, \psi_3^r, \psi_4^r; \psi_1^i, \psi_2^i, \psi_3^i, \psi_4^i); \tag{64}$$

and the matrices Γ^μ are determined as follows

$$\Gamma^\mu = I_8 \otimes \gamma^\mu. \qquad 5)$$

The above performed study may be repeated again with minor modifications; in this way we can verify that the intrinsic symmetry transformations are determined by 28 generators to which there correspond imaginary parameters.

Let us detail the one triple of generators

$$J_1 = i\gamma^1 \otimes I_8, \quad J_3 = i\gamma^3 \otimes I_8, \quad J_{11} = i(\gamma^3\gamma^1) \otimes I_8, \tag{66}$$

they obey the $su(2)$ algebra. The corresponding finite transformations read

$$\Psi' = Q\Psi, \quad Q = \omega_0 I + \omega_1 J_1 + \omega_3 J_3 + \omega_{11} J_{11}, \quad I = I_{32\times32}; \tag{67}$$

where parameter ω_0 is real-valued, and $\omega_1, \omega_3, \omega_{11}$ are imaginary. In initial components of the complete wave function (60), it is presented differently

$$\begin{aligned}
\psi'_{11} &= \psi^r_{11} + \psi^i_{11} = -i\omega_{11}\psi^r_{31} + \omega_0\psi^r_{11} - \omega_3\psi^i_{11} - \omega_1\psi^i_{31} - i\omega_{11}\psi^i_{31} + \omega_0\psi^i_{11} - \omega_3\psi^r_{11} - \omega_1\psi^r_{31}, \\
\psi'_{12} &= \psi^r_{12} + \psi^i_{12} = -i\omega_{11}\psi^r_{32} + \omega_0\psi^r_{12} - \omega_3\psi^i_{12} - \omega_1\psi^i_{32} - i\omega_{11}\psi^i_{32} + \omega_0\psi^i_{12} - \omega_3\psi^r_{12} - \omega_1\psi^r_{32}, \\
\psi'_{13} &= \psi^r_{13} + \psi^i_{13} = -i\omega_{11}\psi^r_{33} + \omega_0\psi^r_{13} - \omega_3\psi^i_{13} - \omega_1\psi^i_{33} - i\omega_{11}\psi^i_{33} + \omega_0\psi^i_{13} - \omega_3\psi^r_{13} - \omega_1\psi^r_{33}, \\
\psi'_{14} &= \psi^r_{14} + \psi^i_{14} = -i\omega_{11}\psi^r_{34} + \omega_0\psi^r_{14} - \omega_3\psi^i_{14} - \omega_1\psi^i_{34} - i\omega_{11}\psi^i_{34} + \omega_0\psi^i_{14} - \omega_3\psi^r_{14} - \omega_1\psi^r_{34},
\end{aligned} \tag{68}$$

$$\begin{aligned}
\psi'_{21} &= \psi^r_{21} + \psi^i_{21} = -i\omega_{11}\psi^r_{41} + \omega_0\psi^r_{21} - \omega_3\psi^i_{21} - \omega_1\psi^i_{41} - i\omega_{11}\psi^i_{41} + \omega_0\psi^i_{21} - \omega_3\psi^r_{21} - \omega_1\psi^r_{41}, \\
\psi'_{22} &= \psi^r_{22} + \psi^i_{22} = -i\omega_{11}\psi^r_{42} + \omega_0\psi^r_{22} - \omega_3\psi^i_{22} - \omega_1\psi^i_{42} - i\omega_{11}\psi^i_{42} + \omega_0\psi^i_{22} - \omega_3\psi^r_{22} - \omega_1\psi^r_{42}, \\
\psi'_{23} &= \psi^r_{23} + \psi^i_{23} = -i\omega_{11}\psi^r_{43} + \omega_0\psi^r_{23} - \omega_3\psi^i_{23} - \omega_1\psi^i_{43} - i\omega_{11}\psi^i_{43} + \omega_0\psi^i_{23} - \omega_3\psi^r_{23} - \omega_1\psi^r_{43}, \\
\psi'_{24} &= \psi^r_{24} + \psi^i_{24} = -i\omega_{11}\psi^r_{44} + \omega_0\psi^r_{24} - \omega_3\psi^i_{24} - \omega_1\psi^i_{44} - i\omega_{11}\psi^i_{44} + \omega_0\psi^i_{24} - \omega_3\psi^r_{24} - \omega_1\psi^r_{44},
\end{aligned} \tag{69}$$

$$\begin{aligned}
\psi'_{31} &= \psi^r_{31} + \psi^i_{31} = i\omega_{11}\psi^r_{11} + \omega_0\psi^r_{31} + \omega_3\psi^i_{31} - \omega_1\psi^i_{11} + i\omega_{11}\psi^i_{11} + \omega_0\psi^i_{31} + \omega_3\psi^r_{31} - \omega_1\psi^r_{11},\\
\psi'_{32} &= \psi^r_{32} + \psi^i_{32} = i\omega_{11}\psi^r_{12} + \omega_0\psi^r_{32} + \omega_3\psi^i_{32} - \omega_1\psi^i_{12} + i\omega_{11}\psi^i_{12} + \omega_0\psi^i_{32} + \omega_3\psi^r_{32} - \omega_1\psi^r_{12},\\
\psi'_{33} &= \psi^r_{33} + \psi^i_{33} = i\omega_{11}\psi^r_{13} + \omega_0\psi^r_{33} + \omega_3\psi^i_{33} - \omega_1\psi^i_{13} + i\omega_{11}\psi^i_{13} + \omega_0\psi^i_{33} + \omega_3\psi^r_{33} - \omega_1\psi^r_{13},\\
\psi'_{34} &= \psi^r_{34} + \psi^i_{34} = i\omega_{11}\psi^r_{14} + \omega_0\psi^r_{34} + \omega_3\psi^i_{34} - \omega_1\psi^i_{14} + i\omega_{11}\psi^i_{14} + \omega_0\psi^i_{34} + \omega_3\psi^r_{34} - \omega_1\psi^r_{14},
\end{aligned} \tag{70}$$

$$\begin{aligned}
\psi'_{41} &= \psi^r_{41} + \psi^i_{41} = i\omega_{11}\psi^r_{21} + \omega_0\psi^r_{41} + \omega_3\psi^i_{41} - \omega_1\psi^i_{21} + i\omega_{11}\psi^i_{21} + \omega_0\psi^i_{41} + \omega_3\psi^r_{41} - \omega_1\psi^r_{21},\\
\psi'_{42} &= \psi^r_{42} + \psi^i_{42} = i\omega_{11}\psi^r_{22} + \omega_0\psi^r_{42} + \omega_3\psi^i_{42} - \omega_1\psi^i_{22} + i\omega_{11}\psi^i_{22} + \omega_0\psi^i_{42} + \omega_3\psi^r_{42} - \omega_1\psi^r_{22},\\
\psi'_{43} &= \psi^r_{43} + \psi^i_{43} = i\omega_{11}\psi^r_{23} + \omega_0\psi^r_{43} + \omega_3\psi^i_{43} - \omega_1\psi^i_{23} + i\omega_{11}\psi^i_{23} + \omega_0\psi^i_{43} + \omega_3\psi^r_{43} - \omega_1\psi^r_{23},\\
\psi'_{44} &= \psi^r_{44} + \psi^i_{44} = i\omega_{11}\psi^r_{24} + \omega_0\psi^r_{44} + \omega_3\psi^i_{44} - \omega_1\psi^i_{24} + i\omega_{11}\psi^i_{24} + \omega_0\psi^i_{44} + \omega_3\psi^r_{44} - \omega_1\psi^r_{24}.
\end{aligned} \tag{71}$$

We may rewrite these formulas in a more symmetrical form

$$\psi'_{11} = (\omega_0 - \omega_3)\psi^r_{11} + (\omega_0 - \omega_3)\psi^i_{11} - (\omega_1 + i\omega_{11})\psi^r_{31} - (\omega_1 + i\omega_{11})\psi^i_{31} =$$
$$= (\omega_0 - \omega_3)(\psi^r_{11} + \psi^i_{11}) - (\omega_1 + i\omega_{11})(\psi^r_{31} + \psi^i_{31}),$$

whence, bearing in mind $\psi^r_{11} + \psi^i_{11} = \psi_{11}, \psi^r_{31} + \psi^i_{31} = \psi_{31}$, we obtain

$$\psi'_{11} = (\omega_0 - \omega_3)\psi_{11} - (\omega_1 + i\omega_{11})\psi_{31}.$$

Similar expressions we get for the variables $\psi'_{12}, \psi'_{13}, \psi'_{14}$:

$$\begin{aligned}
\psi'_{12} &= (\omega_0 - \omega_3)\psi_{12} - (\omega_1 + i\omega_{11})\psi_{32},\\
\psi'_{13} &= (\omega_0 - \omega_3)\psi_{13} - (\omega_1 + i\omega_{11})\psi_{33},\\
\psi'_{14} &= (\omega_0 - \omega_3)\psi_{14} - (\omega_1 + i\omega_{11})\psi_{34}.
\end{aligned}$$

The last 4 relations may be written with the use of one formula (we add similar formulas for remaining columns)

$$\begin{aligned}
\psi'_{1k} &= (\omega_0 - \omega_3)\psi_{1k} - (\omega_1 + i\omega_{11})\psi_{3k},\\
\psi'_{2k} &= (\omega_0 - \omega_3)\psi_{2k} - (\omega_1 + i\omega_{11})\psi_{4k},\\
\psi'_{3k} &= (\omega_0 + \omega_3)\psi_{3k} + (\omega_1 - i\omega_{11})\psi_{1k},\\
\psi'_{4k} &= (\omega_0 + \omega_3)\psi_{4k} + (\omega_1 - i\omega_{11})\psi_{2k}.
\end{aligned} \tag{72}$$

Thus, for four Dirac fields (60), the symmetry transformations (67) are written as follows

$$\begin{vmatrix} \Psi'_{1k} \\ \Psi'_{2k} \\ \Psi'_{3k} \\ \Psi'_{4k} \end{vmatrix} = \begin{vmatrix} (\omega_0 - \omega_3) & 0 & (\omega_1 - i\omega_{11})^* & 0 \\ 0 & (\omega_0 - \omega_3) & 0 & (\omega_1 - i\omega_{11})^* \\ (\omega_1 - i\omega_{11}) & 0 & (\omega_0 - \omega_3)^* & 0 \\ 0 & (\omega_1 - i\omega_{11}) & 0 & (\omega_0 - \omega_3)^* \end{vmatrix} \begin{vmatrix} \Psi_{1k} \\ \Psi_{2k} \\ \Psi_{3k} \\ \Psi_{4k} \end{vmatrix}.$$

This transformation $\Psi' = Q\Psi$ may be decomposed in the linear combination of 4×4 matrices (recall that $\gamma^5 = \gamma^1\gamma^2\gamma^3\gamma^0$)

$$Q=\omega_0 I_4+\omega_3\left|\begin{matrix}-1&&&\\&-1&&\\&&1&\\&&&1\end{matrix}\right|+\omega_1\left|\begin{matrix}&&-1&\\&&&-1\\1&&&\\&1&&\end{matrix}\right|+$$
$$+i\omega_{11}\left|\begin{matrix}&&-1&\\&&&-1\\-1&&&\\&-1&&\end{matrix}\right|=\omega_0 I_4+i\omega_3\gamma^2-\omega_1\gamma^5-\omega_{11}\gamma^2\gamma^5. \qquad (73)$$

Let us introduce new notations

$$\Omega_1=2\omega_3,\quad \Omega_2=-2\omega_1,\quad \Omega_3=-2\omega_{11},$$
$$G_1=\frac{i}{2}\gamma^2,\quad G_2=\frac{1}{2}\gamma^5,\quad G3=\frac{1}{2}\gamma^2\gamma^5,$$

then the Q transformation is presented as follows

$$Q=\omega_0 I_4+i\omega_3\gamma^2-\omega_1\gamma^5-\omega_{11}\gamma^2\gamma^5=\omega_0 I_4+\Omega_1 G_1+\Omega_2 G_2+\Omega_3 G_3.$$

The generators G_i obey the following commutative relations

$$[G_1,G_2]_-=iG_3,\quad [G_3,G_1]_-=iG_2,\quad [G_2,G_3]_-=-iG_1; \qquad (74)$$

in modified notations, $G_1\to K_3, G_2\to K_1, G_3\to K_2$, they read

$$[K_1,K_2]_-=-iK_3,\quad [K_2,K_3]_-=iK_1,\quad [K_3,K_1]_-=iK_2, \qquad (75)$$

which refers to $su(1,1)$ algebra.

Let us consider relations between the four Dirac fields and the Dirac – Kähler theory. As known, the Dirac – Kähler particle may be described in terms of tensor variables $(\varphi,\varphi_i,\tilde{\varphi},\tilde{\varphi}_i,\varphi_{mn})$, and the corresponding equations read [32]

$$\partial_l\varphi+m\varphi_l=0,\quad \partial_l\tilde{\varphi}+m\tilde{\varphi}_l=0,$$
$$\partial_l\varphi+\partial_a\varphi_{la}-m\varphi_l=0,\quad \partial_l\tilde{\varphi}-\frac{1}{2}\varepsilon_l{}^{amn}\partial_a\varphi_{mn}-m\tilde{\varphi}_l=0, \qquad (76)$$
$$\partial_m\varphi_n-\partial_n\varphi_m+\varepsilon_{mn}{}^{ab}\partial_a\tilde{\varphi}_b-m\varphi_{mn}=0;$$

pseudotensor quantities are marked by tilde. Equivalent description is possible with the use of the 2-rank bispinor U, then the system of equations is presented as follows

$$(i\gamma^\mu\partial_\mu-m)U=0, \qquad (77)$$

were we assume the use of the Dirac matrices γ^μ in spinor basis

$$\gamma^0=\left|\begin{matrix}&I_2\\I_2&\end{matrix}\right|,\gamma^1=\left|\begin{matrix}&-\sigma_1\\\sigma_1&\end{matrix}\right|,\gamma^2=\left|\begin{matrix}&-\sigma_2\\\sigma_2&\end{matrix}\right|,\gamma^3=\left|\begin{matrix}&-\sigma_3\\\sigma_3&\end{matrix}\right|.$$

Relations between two sets of variables are determined by the formulas [55]:

$$\varphi = -\frac{1}{4i}\mathrm{Sp}(EU), \quad \varphi_l = \frac{1}{4}\mathrm{Sp}(E\gamma_l U), \quad \tilde{\varphi} = \frac{1}{4}\mathrm{Sp}(E\gamma_5 U),$$
$$\tilde{\varphi}_l = \frac{1}{4i}\mathrm{Sp}(E\gamma_5\gamma_l U), \quad \varphi_{mn} = -\frac{1}{2i}\mathrm{Sp}(E\sigma_{mn}U), \tag{78}$$

where

$$E = \begin{vmatrix} i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{vmatrix}, \quad \sigma^{ab} = \frac{1}{4}(\gamma^a\gamma^b - \gamma^b\gamma^a). \tag{79}$$

Let us transform relations (78) to Majorana basis:

$$\psi_s = S\psi_m, \quad \gamma_s^a = S\gamma_m^a S^{-1},$$
$$S = \frac{1}{2}\begin{vmatrix} i & -i & 1 & -1 \\ i & i & 1 & 1 \\ -i & -i & 1 & 1 \\ i & -i & -1 & 1 \end{vmatrix}, S^{-1} = S^{+} = \frac{1}{2}\begin{vmatrix} -i & -i & i & -i \\ i & -i & i & i \\ 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 \end{vmatrix}. \tag{80}$$

The transformation rule is

$$U = (S \otimes S)U' = SU'S^T, \quad U' = (S^{-1} \otimes S^{-1})U = S^{-1}U'(S^{-1})^T.$$

Further we find explicit expressions for scalar components

$$\varphi' = -\frac{1}{4i}\mathrm{Sp}(SE'S^{-1}SU'S^T) = -\frac{1}{4i}\mathrm{Sp}(SE'U'S^T), \tag{81}$$

where $E' = S^{-1}ES$. Similarly, we get expressions for remaining tensor components

$$\varphi_l' = \frac{1}{4}\mathrm{Sp}(SE'\gamma_l' U'S^T), \quad \tilde{\varphi}' = \frac{1}{4}\mathrm{Sp}(SE'\gamma_5' U'S^T),$$
$$\tilde{\varphi}_l' = \frac{1}{4i}\mathrm{Sp}(SE'\gamma_{5'}\gamma_l' U'S^T), \quad \varphi_{mn}' = -\frac{1}{2i}\mathrm{Sp}(SE'\sigma_{mn}' U'S^T). \tag{82}$$

We consider the matrix U' as a set of 4 columns (These columns do not behave as bispinors under the Lorentz group)

$$U' = (\Psi_{1k}', \Psi_{2k}', \Psi_{3k}', \Psi_{4k}'); \tag{83}$$

the prime indicates on Majorana basis. Because the Dirac – Kähler equation may be presented a set of 4 unlinked Dirac-like equations (however it is not so in Riemannian space-time), the intrinsic symmetries for four Dirac fields should be the symmetries for Dirac – Kähler field as well.

Intrinsic symmetry transformation $\bar{\Psi}' = Q\Psi'$ explicitly reads

$$\bar{\Psi}_1' = (\omega_0 - \omega_3)\Psi_1' + (\omega_1 - i\omega_{11})^*\Psi_3',$$
$$\bar{\Psi}_2' = (\omega_0 - \omega_3)\Psi_2' + (\omega_1 - i\omega_{11})^*\Psi_4',$$
$$\bar{\Psi}_3' = (\omega_0 - \omega_3)^*\Psi_3' + (\omega_1 - i\omega_{11})\Psi_1',$$
$$\bar{\Psi}_4' = (\omega_0 - \omega_3)^*\Psi_4' + (\omega_1 - i\omega_{11})\Psi_2'. \tag{84}$$

Corresponding tensor components are determined by relations

$$
\begin{gathered}
\bar{\varphi}' = -\frac{1}{4i}\,\mathrm{Sp}(SE'\bar{U}'S^T),\\
\bar{\varphi}_l' = \frac{1}{4}\,\mathrm{Sp}(SE'\gamma_l'\bar{U}'S^T), \quad \tilde{\bar{\varphi}}' = \frac{1}{4}\,\mathrm{Sp}(SE'\gamma_5'\bar{U}'S^T),\\
\tilde{\bar{\varphi}}_l' = \frac{1}{4i}\,\mathrm{Sp}(SE'\gamma_{5'}\gamma_l'\bar{U}'S^T), \quad \bar{\varphi}_{mn}' = -\frac{1}{2i}\,\mathrm{Sp}(SE'\sigma_{mn}'\bar{U}'S^T).
\end{gathered} \tag{85}
$$

For example, let us write down the relevant formulas for pseudo-quantities

$$
\begin{aligned}
\bar{\varphi}' = \frac{1}{4i}\{&\omega_0[(\psi_{43}'-\psi_{34}')+(\psi_{12}'-\psi_{21}')]+\omega_3[(\psi_{43}'-\psi_{34}')-(\psi_{12}'-\psi_{21}')]+\\
&+\omega_-(\psi_{23}'-\psi_{14}')+\omega_+(\psi_{41}'-\psi_{32}')\},\\
\bar{\varphi}_1' = -\frac{1}{4i}[&(\omega_0+\omega_3)(\psi_{42}'-\psi_{31}')+(\omega_0-\omega_3)(\psi_{13}'-\psi_{24}')+\\
&+\omega_-(\psi_{14}'+\psi_{22}')+\omega_+(\psi_{44}'-\psi_{31}')],\\
\bar{\varphi}_2' = -\frac{1}{4i}[&(\omega_0-\omega_3)(\psi_{12}'-\psi_{21}')+(\omega_0+\omega_3)(\psi_{34}'-\psi_{43}')-\\
&-\omega_+(\psi_{32}'-\psi_{41}')+\omega_-(\psi_{14}'-\psi_{23}')],\\
\bar{\varphi}_3' = -\frac{1}{4i}[&(\omega_0+\omega_3)(\psi_{32}'+\psi_{41}')-(\omega_0-\omega_3)(\psi_{23}'+\psi_{14}')+\\
&+\omega_-(\psi_{12}'+\psi_{21}')+\omega_+(\psi_{43}'+\psi_{34}')],\\
\bar{\varphi}_0' = -\frac{1}{4i}[&(\omega_0-\omega_3)(\psi_{24}'+\psi_{13}')-(\omega_0+\omega_3)(\psi_{42}'+\psi_{31}')-\\
&-\omega_+(\psi_{44}'+\psi_{33}')+\omega_-(\psi_{22}'-\psi_{11}')].
\end{aligned}
$$

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