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Stueckelberg – Ogievetsky – Polubarinov – Calb – Ramond – Maxwell Fields, the Gauge Degrees of Freedom

A comparative analysis for the problem of the gauge degrees of freedom for massless particles is described by three different systems of equations: Stückelberg's, Ogievetsky – Polubarinov – Kalb – Ramon's, and Maxwell's. All three systems of equations are represented in a unified matrix form, these equations are solved in the Cartesian coordinates. Solutions of the plane wave type are constructed explicitly; correspondingly, we find 5, 4 and 4 independent solutions. In order to decide in each case, which of the solutions correspond to physically observable states and which – to gauge states, we find for all three the matrix of invariant bilinear form, which permits us to fix the structure of the energy-momentum tensor. Expressions for the energy-momentum tensor are obtained in explicit form for all independent solutions (5, 4 and 4). It is shown that for the Stueckelberg field, only one solution corresponds to a state with non-zero energy density, it describes the physically observable state; for 4 remaining solutions the energy-momentum tensor turns out to be zero, which indicates the gauge nature of these solutions. For the Ogievetsky – Polubarinov – Kalb – Ramond field, only one solution turns out to be physically observable. Finally, in the case of the Maxwell field, two solutions are physically observable, and two other solutions are gauge ones. Besides, among the two gauge solutions one has the structure of the ordinary plane wave, and the other describes an unusual plane wave: for it there is no standard relationship between energy and the three components of the linear momentum.

Key Words: *Stueckelberg, Ogievetsky – Polubarinov – Calb – Ramond, Maxwell fields, Cartesian coordinates, exact solutions, energy-momentum tensor, gauge degrees of freedom .*

ПОЛЯ ШТЮКЕЛЬБЕРГА – ОГИЕВЕЦКОГО – ПОЛУБАРИНОВА – КАЛЬБО – РАМОНА – МАКСВЕЛЛА, КАЛИБРОВОЧНЫЕ СТЕПЕНИ СВОБОДЫ

Проведен сопоставительный анализ вопроса о калибровочных степенях свободы для безмассовых частиц, описываемых тремя разными системами уравненний: Штюкельберга, Огиевецкого – Полубаринова – Кальба – Рамона и Максвелла. Все три системы уравнений представляются в унифицированной матричной форме, эти уравнения решены в декартовой системе координат. В явном виде построены решения типа плоских волн; при этом возникают соотвественно 5, 4 и 4 независимых решений. Для того, чтобы решить в каждом случае вопрос, какие из решений отвечают физически наблюдаемым состояниями и какие – калибровочным, во всех трех моделях находятся матрицы инвариантной билинейной фоме и затем структура тензора энергии-импульса. Выражения для тензора энергии –

импульса получены в явном виде для всех независимых решений (5-ти, 4-х и 4-х). Показано, что в случае поля Штюкельберга из 5 решений только одно соответствует состоянию с ненулевой плотностью энергии, оно описывает наблюдаемое состояние; для 4-х остальных решений тензор энергии-импулса оказывается нулевым, что указывает на калибровочных характер этих решений. В случае поля Огиевецкого – Полубаринова – Кальба – Рамона из 4-х найденных решений только одно оказывается физически наблюдаемым. Наконец, в случае поля Максвелла 2 решения являются физически наблюдаемыми, и 2 – калибровочными; причем из двух калибровочных решений одно имеет структуру обычной плоской волны, а второе описывает необычную плоскую волну: для нее не выполняется стандартное соотношение между энергией и тремя компонентами импульса.

Ключевые слова: *Поля Штюкельберга, Огиевецкого – Полубаринова – Кальба – Рамона, Максвелла, декартовые координаты, точные решения, тензор энергии – импульса, калибровочные степени свободы.*

1. Massless Stueckelberg Field

In this section we examine the massless Stueckelberg field. Among 11 components field function, antisymmetric tensor represents the gauge variables, the scalar and vector correspond to physically observable quantities.

It is shown that in Cartesian coordinates the Stueckelberg equations permits existence of five independent solutions describe the different states of the field.

We have derived expression for the energy-momentum tensor of the massless Stueckelberg field. We have found its explicit form for arbitrary linear combination of 5 established solutions. We have found 4 combinations of solutions which do not contribute to energy-momentum tensor, therefore they correspond to purely gauge states. There exists only one solution to which there corresponds nonvanishing energy-momentum tensor, it relates to physically observably state of the massless Stueckelberg field.

1.1. Basic equations

Let us consider the equations for massless Stueckelberg field in Cartesian coordinates [1–4]:

$$\partial^a \Psi_a = 0, \partial_a \Psi + \partial^b \Psi_{ab} - \Psi_a = 0, \partial_a \Psi_b - \partial_b \Psi_a = 0; \qquad (1)$$

as it should be, in this system there is no parameter with the dimension of the inverse length L^{-1}, we apply the metrical tensor with the signature $(+---)$. The physical dimensions of the components obey the rule $[L^{-1}\Psi] = [L^{-1}\Psi_{ab}] = [\Psi_a]$.

We will apply the matrix form of equations (1). As a field function, the 11-dimensional column is used

$$\Phi = (\Psi; \Psi_0, \Psi_1, \Psi_2, \Psi_3; \Psi_{01}, \Psi_{02}, \Psi_{03}, \Psi_{23}, \Psi_{31}, \Psi_{12}) = (H, H_1, H_2). \qquad (2)$$

The system (1) can be written in the block form

$$G^a \partial_a H_1 = 0, \quad \Delta^a \partial_a H + K^a \partial_a H_2 - H_1 = 0, \quad L^a \partial_a H_1 = 0; \qquad (3)$$

in 11-dimensional presentation it reads

$$(\Gamma^a \partial_a - P)\Phi = 0, \Gamma^a = \begin{vmatrix} 0 & G^a & 0 \\ \Delta^a & 0 & K^a \\ 0 & L^a & 0 \end{vmatrix}, \quad P = \begin{vmatrix} 0 & 0 & 0 \\ 0 & I_{4\times 4} & 0 \\ 0 & 0 & 0 \end{vmatrix}, \quad \Phi = \begin{vmatrix} H \\ H_1 \\ H_2 \end{vmatrix}. \qquad (4)$$

Let us write down the explicit form of all matrix blocks:

$$G_0 = (1\,0\,0\,0), G_1 = (0\,-1\,0\,0), G_2 = (0\,0\,-1\,0), G_3 = (0\,0\,0\,-1),$$

$$\Delta^0 = \begin{vmatrix} 1 \\ 0 \\ 0 \\ 0 \end{vmatrix}, \Delta^1 = \begin{vmatrix} 0 \\ 1 \\ 0 \\ 0 \end{vmatrix}, \Delta^2 = \begin{vmatrix} 0 \\ 0 \\ 1 \\ 0 \end{vmatrix}, \Delta^3 = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 1 \end{vmatrix},$$

$$K^0 = \begin{vmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{vmatrix}, K^1 = \begin{vmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{vmatrix},$$

$$K^2 = \begin{vmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 & 0 & 0 \end{vmatrix}, K^3 = \begin{vmatrix} 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix},$$

$$L^0 = \begin{vmatrix} 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}, L^1 = \begin{vmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & +1 & 0 \end{vmatrix}, \quad L^2 = \begin{vmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix}, L^3 = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}.$$

Bearing in mind the substitution for plane wave solutions

$$\Psi(x) = f\varphi(x), \Psi_a = f_a\varphi(x), \Psi_{ab}(x) = f_{ab}(x)\varphi(x); \varphi(x) = e^{-ik_a x^a}; \tag{5}$$

we should make the following changes, $\partial_a \Rightarrow -ik_a, a = 0,1,2,3, k_0 = k^0 = m$. It is convenient to present eq. (4) in the block form

$$\begin{aligned} &(-iG^0k_0 - iG^1k_1 - iG^2k_2 - iG^3k_3)H_1 = 0, \\ &(-i\Delta^a k_a - i\Delta^a k_a - i\Delta^a k_a - i\Delta^a k_a)H \\ &+(-iK^0k_0 - iK^1k_1 - iK^2k_2 - iK^3k_3)H_2 - H_1 = 0, \\ &(-iL^0k_0 - iL^1k_1 - iL^2k_2 - iL^3k_3)H_1 = 0, \end{aligned} \quad H = f, \quad H_1 = \begin{vmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{vmatrix}, \quad H_2 = \begin{vmatrix} E_j \\ B_j \end{vmatrix}, \quad j = 1,2,3. \tag{6}$$

Further we get the algebraic system in explicit matrix form $A\Phi = 0$:

$$\begin{vmatrix} 0 & -k_0 & k_1 & k_2 & k_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ -k_0 & i & 0 & 0 & 0 & k_1 & k_2 & k_3 & 0 & 0 & 0 \\ k_1 & 0 & -i & 0 & 0 & -k_0 & 0 & 0 & 0 & k_3 & -k_2 \\ -k_2 & 0 & 0 & i & 0 & 0 & k_0 & 0 & k_3 & 0 & -k_1 \\ k_3 & 0 & 0 & 0 & -i & 0 & 0 & -k_0 & k_2 & -k_1 & 0 \\ 0 & k_1 & -k_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & k_2 & 0 & -k_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & k_3 & 0 & 0 & -k_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & k_3 & -k_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -k_3 & 0 & k_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & k_2 & -k_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix} \begin{vmatrix} f \\ f_0 \\ f_1 \\ f_2 \\ f_3 \\ E_1 \\ E_2 \\ E_3 \\ B_1 \\ B_2 \\ B_3 \end{vmatrix} = 0. \qquad (7)$$

Rank of the matrix equals 8. Allowing for the known condition $k_0 = \sqrt{k_1^2 + k_2^2 + k_3^2}$, we get the matrix of the rank 6.

Removing the fifth and four last rows, we get the matrix with the same rank

$$\begin{vmatrix} 0 & -k_0 & k_1 & k_2 & k_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ -k_0 & i & 0 & 0 & 0 & k_1 & k_2 & k_3 & 0 & 0 & 0 \\ k_1 & 0 & -i & 0 & 0 & -k_0 & 0 & 0 & 0 & k_3 & -k_2 \\ -k_2 & 0 & 0 & i & 0 & 0 & k_0 & 0 & k_3 & 0 & -k_1 \\ 0 & k_1 & -k_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & k_2 & 0 & -k_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix} \begin{vmatrix} f \\ f_0 \\ f_1 \\ f_2 \\ f_3 \\ E_1 \\ E_2 \\ E_3 \\ B_1 \\ B_2 \\ B_3 \end{vmatrix} = 0. \qquad (8)$$

As a result, we arrive at the inhomogeneous system

$$\begin{vmatrix} 0 & -k_0 & k_1 & k_2 & 0 & 0 \\ -k_0 & i & 0 & 0 & k_1 & k_2 \\ k_1 & 0 & -i & 0 & -k_0 & 0 \\ -k_2 & 0 & 0 & i & 0 & k_0 \\ 0 & k_1 & -k_0 & 0 & 0 & 0 \\ 0 & k_2 & 0 & -k_0 & 0 & 0 \end{vmatrix} \begin{vmatrix} f \\ f_0 \\ f_1 \\ f_2 \\ E_1 \\ E_2 \end{vmatrix} = -\begin{vmatrix} k_3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{vmatrix} f_3 - \begin{vmatrix} 0 \\ k_3 \\ 0 \\ 0 \\ 0 \\ 0 \end{vmatrix} E_3 - \begin{vmatrix} 0 \\ 0 \\ 0 \\ k_3 \\ 0 \\ 0 \end{vmatrix} B_1 - \begin{vmatrix} 0 \\ 0 \\ k_3 \\ 0 \\ 0 \\ 0 \end{vmatrix} B_2 - \begin{vmatrix} 0 \\ 0 \\ -k_2 \\ -k_1 \\ 0 \\ 0 \end{vmatrix} B_3; \qquad (9)$$

the determinant of the matrix is $\det A_{6\times 6} = k_3^4 \left(k_1^2 + k_2^2 + k_3^2 \right)$.

Further we find 5 independent solutions:

$$\Psi_1 = \begin{vmatrix} f \\ f_0 \\ f_1 \\ f_2 \\ E_1 \\ E_2 \end{vmatrix} = \begin{vmatrix} i/k_3 \\ k_0/k_3 \\ k_1/k_3 \\ k_2/k_3 \\ 0 \\ 0 \end{vmatrix} f_3, \; f \neq 0, \; f_0 \neq 0, \; f_i \neq 0, \; E_i = 0, \; B_i = 0,$$

all gauge variables vanish, so this solution may be considered as the physical one Ψ_1;

$$\Psi_2 = \begin{vmatrix} f \\ f_0 \\ f_1 \\ f_2 \\ E_1 \\ E_2 \end{vmatrix} = \begin{vmatrix} k_0/k_3 \\ 0 \\ 0 \\ 0 \\ k_1/k_3 \\ k_2/k_3 \end{vmatrix} E_3, \; f \neq 0, \; f_0 = 0, \; f_i = 0, \; E_i \neq 0, \; B_i = 0,$$

observable and gauge variables are presented here;

$$\Psi_3 = \begin{vmatrix} f \\ f_0 \\ f_1 \\ f_2 \\ E_1 \\ E_2 \end{vmatrix} = \begin{vmatrix} -k_2/k_3 \\ 0 \\ 0 \\ 0 \\ -k_1k_2/k_3k_0 \\ -(k_2^2+k_3^2)/k_3k_0 \end{vmatrix} B_1, \quad f \neq 0, \; f_0 = 0, \; f_i = 0, E_{1,2} \neq 0, E_3 = 0, B_1 \neq 0, B_{2,3} = 0,$$

observable and gauge variables are presented here;

$$\Psi_4 = \begin{vmatrix} f \\ f_0 \\ f_1 \\ f_2 \\ E_1 \\ E_2 \end{vmatrix} = \begin{vmatrix} k_1/k_3 \\ 0 \\ 0 \\ 0 \\ (k_1^2+k_3^2)/k_0k_3 \\ k_1k_2/k_0k_3 \end{vmatrix} B_2, \quad f \neq 0, \; f_0 = 0, \; f_i = 0, E_{1,2} \neq 0, E_3 = 0, B_2 \neq 0, B_{1,3} = 0,$$

observable and gauge variables are presented here;

$$\Psi_5^{gauge} = \begin{vmatrix} f \\ f_0 \\ f_1 \\ f_2 \\ E_1 \\ E_2 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -k_2/k_0 \\ k_1/k_0 \end{vmatrix} B_3, \quad f = 0, \; f_0 = 0, \; f_i = 0, \; dE_{1,2} \neq 0, E_3 = 0, B_{1,2} = 0, B_3 \neq 0;$$

only gauge variables are presented here, so this solution might be considered as purely gauge as well.

In 11-dimensional form, these 5 solutions read

$$\Psi_1 = \begin{vmatrix} i/k_3 \\ k_0/k_3 \\ k_1/k_3 \\ k_2/k_3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{vmatrix} f_3, \Psi_5^{gauge} = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -k_2/k_0 \\ k_1/k_0 \\ 0 \\ 0 \\ 0 \\ 1 \end{vmatrix} B_3, \Psi_2 = \begin{vmatrix} k_0/k_3 \\ 0 \\ 0 \\ 0 \\ 0 \\ k_1/k_3 \\ k_2/k_3 \\ 1 \\ 0 \\ 0 \\ 0 \end{vmatrix} E_3, \Psi_3 = \begin{vmatrix} -k_2/k_3 \\ 0 \\ 0 \\ 0 \\ 0 \\ -k_1k_2/(k_3k_0) \\ -(k_2^2+k_3^2)/(k_3k_0) \\ 0 \\ 1 \\ 0 \\ 0 \end{vmatrix} B_1, \Psi_4 = \begin{vmatrix} k_1/k_3 \\ 0 \\ 0 \\ 0 \\ 0 \\ (k_1^2+k_3^2)/(k_3k_0) \\ (k_1k_2)/(k_3k_0) \\ 0 \\ 0 \\ 1 \\ 0 \end{vmatrix} B_2. \qquad (10)$$

It is necessary to have known, which solutions are the gauge and which are physically observable.

1.2. Energy-Momentum Tensor, the Gauge Solutions

In order to separate the gauge solutions from physically observable ones, we should specify the structure of the energy-momentum tensor for the Stueckelberg field. To this end, we start with the equation

$$(\Gamma^a \partial_a - P)\Psi = 0. \qquad (11)$$

Let us introduce the conjugated function (η is the matrix of the bilinear form)

$$\bar{\Psi} = \Psi^+ \eta, \ \Psi^+ \eta\eta[(\Gamma^a)^+ \overleftarrow{\partial}_a - P]\eta = 0, \ \eta^{-1} = 1,$$

or differently

$$\Psi^+ \eta = \bar{\Psi}, \ \bar{\Psi}[\eta(\Gamma^a)^+ \eta \overleftarrow{\partial}_a - \eta P \eta] = 0.$$

Let us impose the following constraints

$$\eta(\Gamma^a)^+ \eta = -\Gamma^a, \quad \eta P \eta = P, \quad \eta^2 = 1; \qquad (12)$$

relations (12) may be presented as follows

$$\eta(\Gamma^a)^+ = -\Gamma^a \eta, \quad \eta P = P\eta, \ \eta^2 = 1. \qquad (13)$$

Then the conjugated equation reads

$$\bar{\Psi}(\Gamma^a \overleftarrow{\partial}_a + P) = 0. \qquad (14)$$

Multiplying eq. (11) from the left by $\bar{\Psi}$, and multiplying eq. (14) from the right by Ψ, and summing the results we get

$$\bar{\Psi}\Gamma^a \partial_a \Psi + \bar{\Psi}\Gamma^a \overleftarrow{\partial}_a \Psi = 0 \quad \Rightarrow \quad \partial_a(\bar{\Psi}\Gamma^a \Psi) = 0.$$

This is the conservation law for the current $J^a = \bar{\Psi}\Gamma^a\Psi$. Let us define the energy-momentum tensor.

To this end, we act on eq. (11) by operator $\bar{\Psi}\partial_b$, and multiply eq. (14) from the right by $\partial_b\Psi$, and sum the results

$$\bar{\Psi}\partial_b(\Gamma^a\partial_a - P)\Psi + \bar{\Psi}(\Gamma^a \overleftarrow{\partial}_a + P)\partial_b\Psi = 0.$$

Further we derive

$$\bar{\Psi}\partial_a\Gamma^a\partial_b\Psi + (\partial_a\bar{\Psi})\Gamma^a\partial_b\Psi = 0,$$

this gives

$$\partial_a(\bar{\Psi}(x)\Gamma^a\partial_b\Psi(x)) = 0,\ R^a{}_b(x) = \bar{\Psi}(x)\Gamma^a\partial_b\Psi(x),\ \partial_a R^a{}_b(x) = 0. \tag{15}$$

Let us find explicit form of the matrix η, it should obey the following restrictions

$$h^{-1}(\Gamma^a)^+\eta = -\Gamma^a \quad \Rightarrow \quad \tilde{\Gamma}^a\eta = -\eta\Gamma^a, \quad \eta = \begin{vmatrix} A_{1\times1} & 0 & 0 \\ 0 & B_{4\times4} & 0 \\ 0 & 0 & C_{6\times6} \end{vmatrix}. \tag{16}$$

The constraint $\tilde{\Gamma}^0\eta + \eta\Gamma^0$ gives

$$\eta = \begin{vmatrix} 1 & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{vmatrix}, B = \begin{vmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}, C = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{vmatrix}. \tag{17}$$

Let us start with the tensor $R^a{}_b$, defined by the formula

$$R^a{}_b(x) = \Psi^+\eta\Gamma^a\partial_b\Psi, \tag{18}$$

that is

$$R^a{}_b(x) = H^+G^a\partial_b H_1 + H_1^+B(\Delta^a\partial_b H + K^a\partial_b H_2) + H_2^+CL^a\partial_b H_1. \tag{19}$$

Taking into account the substitutions in the form of the plane waves, we get

$$R^a{}_b(x) = -i[H^+AG^a k_b H_1 + H_1^+B(\Delta^a k_b H + K^a k_b H_2) + H_2^+CL^a k_b H_1]. \tag{20}$$

Now we calculate this tensor for 5 independent solutions (10). To this end, we are to find three terms in explicit form for each solution:

$$\underline{\Psi_1}, \quad R^a{}_b = \begin{vmatrix} -\frac{2k_0^4}{k_3^2} & -\frac{2k_0^3 k_1}{k_3^2} & -\frac{2k_0^3 k_2}{k_3^2} & -\frac{2k_0^3}{k_3} \\ \frac{2k_0^3 k_1}{k_3^2} & \frac{2k_0^2 k_1^2}{k_3^2} & \frac{2k_0^2 k_1 k_2}{k_3^2} & \frac{2k_0^2 k_1}{k_3} \\ \frac{2k_0^3 k_2}{k_3^2} & \frac{2k_0^2 k_1 k_2}{k_3^2} & \frac{2k_0^2 k_2^2}{k_3^2} & \frac{2k_0^2 k_2}{k_3} \\ \frac{2k_0^3}{k_3} & \frac{2k_0^2 k_1}{k_3} & \frac{2k_0^2 k_2}{k_3} & 2k_0^2 \end{vmatrix} \neq 0; \tag{21}$$

this solution represents physically observable state. We readily verify that four remaining solutions $\Psi_2, \Psi_3, \Psi_4, \Psi_5$ give zero contribution to tensor $R^a{}_b$:

$$\underline{\Psi_i}, \quad R^a{}_b \equiv 0, \quad i = 2,3,4,5; \tag{22}$$

they represent the gauge states.

1.3. Conclusion

It is shown that in Cartesian coordinates, the Stueckelberg massless equation permits existence of five independent solutions describing the different states of the field.

We have derived expression for the energy-momentum tensor of this field and show that this tensor vanishes identically for 4 independent solutions, and does not vanish only for one solution which represents the physically observable state of Stueckelberg massless particle.

Four remaining solutions relate to the gauge states.

2. The Calb – Ramond Field

In this section we examine the Ogievetsky – Polubarinov – Calb – Ramond field [5–7] in Cartesian coordinates.

Among 10 components of the field function, antisymmetric tensor represents the gauge variables, whereas the vector corresponds to physically observable ones.

It is shown that the Calb – Ramond equation permits existence of 4 independent solutions which describe the different states of the field.

We have derived expression for the energy-momentum tensor of the Calb – Ramond field. There exists only one solution to which there corresponds the nonvanishing energy-momentum tensor, it relates to physically observable state of the Calb – Ramond field.

2.1. Basic equation

Let us consider the system of equations for Ogievetsky – Polubarinov – Calb – Ramond field in Cartesian coordinates:

$$\partial^b \Psi_{ab} - \Psi_a = 0, \quad \partial_a \Psi_b - \partial_b \Psi_a = 0; \tag{23}$$

in this system there is no parameter with the dimension of the inverse length.

Tensor components are considered as gauge ones. The physical dimensions of the involved components obey the rule $[(1/L)\Psi_{ab}] = [\Psi_a]$. We will apply the matrix form of equations (23), using the 10-dimensional column

$$\Phi = (\Psi_0, \Psi_1, \Psi_2, \Psi_3;\ \Psi_{01}, \Psi_{02}, \Psi_{03}, \Psi_{23}, \Psi_{31}, \Psi_{12}) = (H_1, H_2). \tag{24}$$

The system (23) can be written in the block form

$$K^a \partial_a H_2 - H_1 = 0, \quad L^a \partial_a H_1 = 0; \tag{25}$$

in 10-dimensional presentation it reads

$$(\Gamma^a \partial_a - P)\Phi = 0, \Gamma^a = \begin{vmatrix} 0 & K^a \\ L^a & 0 \end{vmatrix}, P = \begin{vmatrix} I_{4\times4} & 0 \\ 0 & 0 \end{vmatrix}, \Phi = \begin{vmatrix} H_1 \\ H_2 \end{vmatrix}. \tag{26}$$

Let us write down the explicit form of all matrix blocks

$$K^0 = \begin{vmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{vmatrix}, \quad K^1 = \begin{vmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{vmatrix},$$

$$K^2 = \begin{vmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 & 0 & 0 \end{vmatrix}, \quad K^3 = \begin{vmatrix} 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix},$$

$$L^0 = \begin{vmatrix} 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}, L^1 = \begin{vmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & +1 & 0 \end{vmatrix}, \quad L^2 = \begin{vmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix}, L^3 = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}.$$

Bearing in mind the substitutions for plane waves

$$\Psi_a = f_a \varphi(x), \quad \Psi_{ab} = f_{ab} \varphi(x), \quad \varphi(x) = e^{-ik_a x^a}; \tag{27}$$

we make the changes $\partial_a \Rightarrow -ik_a, a = 0,1,2,3, k_0 = k^0 = m$.

Equation (26) takes the form

$$\begin{aligned} &(-iK^0 k_0 - iK^1 k_1 - iK^2 k_2 - iK^3 k_3) H_2 = H_1, \\ &(-iL^0 k_0 - iL^1 k_1 - iL^2 k_2 - iL^3 k_3) H_1 = 0, \end{aligned} \qquad H_1 = \begin{vmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{vmatrix}, \quad H_2 = \begin{vmatrix} E_j \\ B_j \end{vmatrix}.$$

Further we obtain the algebraic system, we present it in the matrix $A\Phi = 0$:

$$\begin{vmatrix} i & 0 & 0 & 0 & k_1 & k_2 & k_3 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 & -k_0 & 0 & 0 & 0 & k_3 & -k_2 \\ 0 & 0 & i & 0 & 0 & k_0 & 0 & k_3 & 0 & -k_1 \\ 0 & 0 & 0 & -i & 0 & 0 & -k_0 & k_2 & -k_1 & 0 \\ k_1 & -k_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ k_2 & 0 & -k_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ k_3 & 0 & 0 & -k_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & k_3 & -k_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -k_3 & 0 & k_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & k_2 & -k_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix} \begin{vmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \\ E_1 \\ E_2 \\ E_3 \\ B_1 \\ B_2 \\ B_3 \end{vmatrix} = 0. \qquad (28)$$

Acting in the same way as for the case of Stuekelberg field, we find 4 independent solutions:

$$\Phi_1 = \begin{vmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \\ E_1 \\ E_2 \\ E_3 \\ B_1 \\ B_2 \\ B_3 \end{vmatrix}_1 = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -k_3/k_0 \\ k_2/k_0 \\ 1 \\ 0 \\ 0 \end{vmatrix} B_1, \Phi_2 = \begin{vmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \\ E_1 \\ E_2 \\ E_3 \\ B_1 \\ B_2 \\ B_3 \end{vmatrix}_2 = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \\ k_3/k_0 \\ 0 \\ -k_1/k_0 \\ 0 \\ 1 \\ 0 \end{vmatrix} B_2, \Phi_3 = \begin{vmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \\ E_1 \\ E_2 \\ E_3 \\ B_1 \\ B_2 \\ B_3 \end{vmatrix}_3 = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -k_2/k_0 \\ k_1/k_0 \\ 0 \\ 0 \\ 0 \\ 1 \end{vmatrix} B_3, \Phi_4 = \begin{vmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \\ E_1 \\ E_2 \\ E_3 \\ B_1 \\ B_2 \\ B_3 \end{vmatrix}_4 = \begin{vmatrix} 1 \\ k_1/k_0 \\ k_2/k_0 \\ k_3/k_0 \\ -ik_1/k_0^2 \\ -ik_2/k_0^2 \\ -ik_3/k_0^2 \\ 0 \\ 0 \\ 0 \end{vmatrix} f_0. \qquad (29)$$

2.2. The gauge degrees of freedom

Let us turn to additional study of the solutions $\Phi_1, \Phi_2, \Phi_3, \Phi_4$.

By definition, the gauge solutions should not contribute to the energy-momentum tensor of the Ogievetsky – Polubarinov – Calb – Ramond field.

In order to find this tensor, we need the matrix η of the invariant bilinear form:

$$\eta^{-1}(\Gamma^a)^+\eta = -\Gamma^a, \quad \tilde{\Gamma}^a\eta = \eta\Gamma^a, \quad \eta = \begin{vmatrix} B_{4\times4} & 0 \\ 0 & C_{6\times6} \end{vmatrix}; \qquad (30)$$

$$\eta = \begin{vmatrix} B & 0 \\ 0 & C \end{vmatrix}, B = \beta \begin{vmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}, C = t \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{vmatrix};$$

the constraint $\tilde{\Gamma}^0\eta + \eta\Gamma^0 = 0$ gives

$$\eta = \begin{vmatrix} B & 0 \\ 0 & C \end{vmatrix}, \quad B = -\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix}, \quad C = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{vmatrix}. \tag{31}$$

Let us find the energy-momentum tensor for this field. To this end, consider the following quantity:

$$R^a{}_b(x) = H_1^+ \, BK^a \partial_b H_2 + H_2^+ \, CL^a \partial_b H_1. \tag{32}$$

Bearing in mind the substitution in the form of plane waves, from (30) we get

$$R^a{}_b(x) = -i[H_1^+ BK^a k_b H_2 + H_2^+ CL^a k_b H_1]. \tag{33}$$

Further we calculate the tensor $R^a{}_b(x)$ for 4 independent solutions. After simple calculation, we get zeros identically for first 3 solutions Φ_1, Φ_2, Φ_3:

$$(R^a{}_b)_{|\Phi_1} \equiv 0, \quad (R^a{}_b)_{|\Phi_2} \equiv 0, \quad (R^a{}_b)_{|\Phi_3} \equiv 0. \tag{34}$$

For the fourth solution Φ_4, we obtain

$$R^a{}_b = 2\begin{vmatrix} 1 & -\frac{k_1}{k_0} & -\frac{k_2}{k_0} & -\frac{k_3}{k_0} \\ -\frac{k_1}{k_0} & -\frac{k_1^2}{k_0^2} & -\frac{k_1k_2}{k_0^2} & -\frac{k_1k_3}{k_0^2} \\ -\frac{k_2}{k_0} & -\frac{k_1k_2}{k_0^2} & -\frac{k_2^2}{k_0^2} & -\frac{k_2k_3}{k_0^2} \\ -\frac{k_3}{k_0} & -\frac{k_1k_3}{k_0^2} & -\frac{k_2k_3}{k_0^2} & -\frac{k_3^2}{k_0^2} \end{vmatrix} f_0 f_0^* \neq 0. \tag{35}$$

We can varify that the 4-vector of the current for 4 behaves as follows:

$$J^a = \bar{\Psi}\Gamma^a\Psi = \tilde{\Psi}^*\eta\Gamma^a\Psi, \qquad \Psi_4, \;\; J^a = 2i\,\frac{1}{k_0^2}\begin{vmatrix} k_0 \\ -k_1 \\ -k_2 \\ -k_3 \end{vmatrix} f_0^* f_0, \tag{36}$$

for solutions $\Psi_1, ..., \Psi_3$ the current J^a turns out to be vanished.

2.3. Conclusion

In this section, we have shown that for Ogievetsky – Polubarinov – Calb – Ramond field there exists only one solution to which there corresponds nonvanishing energy-momentum tensor, it relates to physically observable state of this field. In contrast to that situation, in Maxwell theory there arise two physical and two gauge states.

3. Maxwell field

In the section, we examine the Maxwell field in Cartesian coordinates. Among 10 components field function, antisymmetric tensor represents the physically observable quantities, whereas the vector corresponds to gauge variables.

It is shown that the Maxwell equations permits existence of 4 independent solutions which describe the different states of the field. We have derived expression for the energy-momentum tensor of the this field.

We find its explicit form for arbitrary linear combination of 4 established solutions. We have found 2 solutions which do not contribute to the energy-momentum tensor, therefore they correspond to purely gauge states.

There exists 2 solutions to which there corresponds the nonvanishing energy-momentum tensor, it relates to physically observable states of the electromagnetic field.

3.1. Cartesian Coordinates

Let us consider the system of equations for Maxwell field in Cartesian coordinates

$$\partial^b \Psi_{ab} = 0, \quad \partial_a \Psi_b - \partial_b \Psi_a = \Psi_{ab}. \qquad (37)$$

The physical dimensions of the involved components are $[(1/L)\Psi_{ab}] = [\Psi_a]$. We will apply the matrix form of equations (37), using the 10-dimensional column

$$\Phi = (\Psi_0, \Psi_1, \Psi_2, \Psi_3;\ \Psi_{01}, \Psi_{02}, \Psi_{03}, \Psi_{23}, \Psi_{31}, \Psi_{12}) = (H_1, H_2). \qquad (38)$$

The system (37) can be written in the block form

$$K^a \partial_a H_2 = 0, \quad L^a \partial_a H_1 = H_2; \qquad (39)$$

in 10-dimensional presentation it reads

$$(\Gamma^a \partial_a - P)\Phi = 0, \qquad \Gamma^a = \begin{vmatrix} 0 & K^a \\ L^a & 0 \end{vmatrix}, \quad P = \begin{vmatrix} 0 & 0 \\ 0 & I_{6\times 6} \end{vmatrix}, \quad \Phi = \begin{vmatrix} H_1 \\ H_2 \end{vmatrix}. \qquad (40)$$

The explicit form of all matrix blocks was given in section 2. Bearing in mind the substitutions for plane waves, we obtain

$$\begin{array}{l} (-iK^0 k_0 - iK^1 k_1 - iK^2 k_2 - iK^3 k_3) H_2 = 0, \\ (-iL^0 k_0 - iL^1 k_1 - iL^2 k_2 - iL^3 k_3) H_1 = H_2, \end{array} \quad H_1 = \begin{vmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{vmatrix}, H_2 = \begin{vmatrix} E_j \\ B_j \end{vmatrix}. \qquad (41)$$

Further we derive the algebraic system in explicit form

$$\begin{vmatrix} 0 & 0 & 0 & 0 & k_1 & k_2 & k_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & k_0 & 0 & 0 & 0 & -k_3 & k_2 \\ 0 & 0 & 0 & 0 & 0 & k_0 & 0 & k_3 & 0 & -k_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & k_0 & -k_2 & k_1 & 0 \\ k_1 & -k_0 & 0 & 0 & i & 0 & 0 & 0 & 0 & 0 \\ k_2 & 0 & -k_0 & 0 & 0 & i & 0 & 0 & 0 & 0 \\ k_3 & 0 & 0 & -k_0 & 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & k_3 & -k_2 & 0 & 0 & 0 & i & 0 & 0 \\ 0 & -k_3 & 0 & k_1 & 0 & 0 & 0 & 0 & i & 0 \\ 0 & k_2 & -k_1 & 0 & 0 & 0 & 0 & 0 & 0 & i \end{vmatrix} \begin{vmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \\ E_1 \\ E_2 \\ E_3 \\ B_1 \\ B_2 \\ B_3 \end{vmatrix} = 0. \tag{42}$$

The rank of the matrix equals 9. We can verify that the rank remains the same if we remove the first row of the matrix.

When removing in $A_{9\times10}$ the first column, we get the square matrix $A_{9\times9}$ with determinant $\det A_{9\times9} = -ik_0^2\left(-k_0^2 + k_1^2 + k_2^2 + k_3^2\right)^2$.

First, let us do not impose the constraint $-k_0^2 + k_1^2 + k_2^2 + k_3^2 = 0$, thereby we search for solution which does not obey the Lorenz condition:

$$\Phi_a = \partial_a \lambda, \quad \partial^a \Phi_a \neq 0, \quad \lambda = e^{-if_0 x_0} e^{-ik_j x_j}; \tag{43}$$

in order to avoid ambiguity, we change the symbol k_0 by f_0.

Correspondingly, the system (42) is equivalent to the following nonhomogeneous equation

$$\begin{vmatrix} 0 & 0 & 0 & f_0 & 0 & 0 & 0 & -k_3 & k_2 \\ 0 & 0 & 0 & 0 & f_0 & 0 & k_3 & 0 & -k_1 \\ 0 & 0 & 0 & 0 & 0 & f_0 & -k_2 & k_1 & 0 \\ -f_0 & 0 & 0 & i & 0 & 0 & 0 & 0 & 0 \\ 0 & -f_0 & 0 & 0 & i & 0 & 0 & 0 & 0 \\ 0 & 0 & -f_0 & 0 & 0 & i & 0 & 0 & 0 \\ 0 & k_3 & -k_2 & 0 & 0 & 0 & i & 0 & 0 \\ -k_3 & 0 & k_1 & 0 & 0 & 0 & 0 & i & 0 \\ k_2 & -k_1 & 0 & 0 & 0 & 0 & 0 & 0 & i \end{vmatrix} \begin{vmatrix} f_1 \\ f_2 \\ f_3 \\ E_1 \\ E_2 \\ E_3 \\ B_1 \\ B_2 \\ B_3 \end{vmatrix} = - \begin{vmatrix} 0 \\ 0 \\ 0 \\ k_1 \\ k_2 \\ k_3 \\ 0 \\ 0 \\ 0 \end{vmatrix} f_0, \quad f_0 \neq k_0. \tag{44}$$

From (44) it follows to the following solution

$$\Phi_{9\times1}=\begin{vmatrix}f_1\\f_2\\f_3\\E_1\\E_2\\E_3\\B_1\\B_2\\B_3\end{vmatrix}=\begin{vmatrix}k_1\\k_2\\k_3\\0\\0\\0\\0\\0\\0\end{vmatrix}\Rightarrow\Phi_{10\times1}=\begin{vmatrix}f_0\\f_1\\f_2\\f_3\\E_1\\E_2\\E_3\\B_1\\B_2\\B_3\end{vmatrix}=\begin{vmatrix}1\\k_1\\k_2\\k_3\\0\\0\\0\\0\\0\\0\end{vmatrix}_0 e^{-i(k_0x_0+k_jx_j)},\ f_0\neq\sqrt{k_1^2+k_2^2+k_3^2}; \qquad (45)$$

it is not the ordinary plane wave, because $f_0 \neq k_0$.

Now let us turn back to the system (42), and take into account the constraint $k_0^2 = k_1^2 + k_2^2 + k_3^2$; so we get the matrix with the rank 7. Removing the rows 3,5,8, we get matrix with the same rank. Further we arrive at the nonhomogeneous equation

$$\begin{vmatrix}0&0&k_1&k_2&k_3&0&0\\0&0&0&k_0&0&k_3&0\\0&0&0&0&k_0&-k_2&k_1\\k_2&0&0&i&0&0&0\\k_3&0&0&0&i&0&0\\0&-k_3&0&0&0&0&i\\0&k_2&0&0&0&0&0\end{vmatrix}\begin{vmatrix}f_0\\f_1\\E_1\\E_2\\E_3\\B_1\\B_2\end{vmatrix}=-\begin{vmatrix}0\\0\\0\\-k_0\\0\\0\\-k_1\end{vmatrix}f_2-\begin{vmatrix}0\\0\\0\\0\\-k_0\\k_1\\0\end{vmatrix}f_3-\begin{vmatrix}0\\-k_1\\0\\0\\0\\0\\i\end{vmatrix}B_3;$$

Let us write down all four solutions in 10-dimensional form ($K(x)=e^{-i(k_0x_0+k_jx_j)}$)

$$\Phi_1^{gauge}=\begin{vmatrix}f_0\\k_1\\k_2\\k_3\\0\\0\\0\\0\\0\\0\end{vmatrix}e^{-i(f_0x_0+k_jx_j)},\quad \Phi_2=\begin{vmatrix}\frac{\left(k_1^2+k_2^2\right)}{k_0k_2}\\\frac{k_1}{k_2}\\1\\0\\-\frac{ik_1k_3^2}{k_0k_2}\\-\frac{ik_3^2}{k_0}\\\frac{i\left(k_3k_1^2+k_2^2k_3\right)}{k_0k_2}\\ik_3\\-\frac{ik_1k_3}{k_2}\\0\end{vmatrix}K\,f_2,\Phi_3=\begin{vmatrix}\frac{k_3}{k_0}\\0\\0\\1\\\frac{ik_1k_3}{k_0}\\\frac{ik_2k_3}{k_0}\\-\frac{i\left(k_1^2+k_2^2\right)}{k_0}\\-ik_2\\ik_1\\0\end{vmatrix}K\,f_3,\quad \Phi_4=\begin{vmatrix}-\frac{ik_1}{k_0k_2}\\-\frac{i}{k_2}\\0\\0\\-\frac{\left(k_2^2+k_3^2\right)}{k_0k_2}\\\frac{k_1}{k_0}\\\frac{k_1k_3}{k_0k_2}\\0\\-\frac{k_3}{k_2}\\1\end{vmatrix}K\,B_3. \qquad (46)$$

Solution Φ_1^{gauge} is pure gauge, because $f_0 \neq \sqrt{k_1^2 + k_2^2 + k_3^2}$ this solution is not a plane wave in the ordinary sense. Solutions 2,3,4 are mixed, they contain both physical and gauge variables; besides, they have the structure of the ordinary plane waves. Let us consider the linear combination of two mixed solutions $a_2\Phi_2 + a_3\Phi_3$; because we wish to construct a solution of the gauge type we will follow only 6 tensor components

$$\varphi_2 = \begin{vmatrix} -ik_1k_3^2/(k_0k_2) \\ -ik_3^2/k_0 \\ i\left(k_3k_1^2 + k_2^2k_3\right)/k_0k_2 \\ ik_3 \\ -ik_1k_3/k_2 \\ 0 \end{vmatrix} f_2, \quad \varphi_3 = \begin{vmatrix} ik_1k_3/k_0 \\ ik_2k_3/k_0 \\ -i\left(k_1^2 + k_2^2\right)/k_0 \\ -ik_2 \\ ik_1 \\ 0 \end{vmatrix} f_3.$$

Let us impose the following condition $a_2\varphi_2 + a_3\varphi_3 = 0$, then the above equation leads to the following system

$$\begin{vmatrix} -(ik_1k_3^2)/(k_2k_0) & ik_1k_3/k_0 \\ -ik_3^2/k_0 & ik_2k_3/k_0 \\ i\left(k_3k_1^2 + k_2^2k_3\right)/k_2k_0 & -i\left(k_1^2 + k_2^2\right)/k_0 \\ ik_3 & -ik_2 \\ -ik_1k_3/k_2 & ik_1 \end{vmatrix} \begin{vmatrix} a_2 f_2 \\ a_3 f_3 \end{vmatrix} = 0.$$

The rank of the matrix equals 1, so we have only one independent equation (let us take the fourth one)

$$ik_3a_2f_2 - ik_2a_3f_3 = 0 \quad \Rightarrow \quad a_2 = \frac{k_2}{f_2}, \quad a_3 = \frac{k_3}{f_3}. \tag{47}$$

Therefore relation $a_2\varphi_2 + a_3\varphi_3 = 0$ leads to

$$\frac{k_2}{f_2}\varphi_2 + \frac{k_3}{f_3}\varphi_3 = 0. \tag{48}$$

Therefore in 10-dimensional representation we have (because solutions Φ_2 and Φ_3 are defined up to arbitrary multiplier, we can set $f_2 = f_3 = f$; also the exponential multiplier is added)

$$a_2\Phi_2 + a_3\Phi_3 = \frac{k_2}{f_2}\begin{vmatrix} \left(k_1^2 + k_2^2\right)/k_0k_2 \\ k_1/k_2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{vmatrix} + \frac{k_3}{f_3}\begin{vmatrix} k_3/k_0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{vmatrix} = \frac{1}{f}\begin{vmatrix} k_0 \\ k_1 \\ k_2 \\ k_3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{vmatrix} e^{-i(k_0x_0 + k_jx_j)} = \Phi_2^{gauge}, \tag{49}$$

evidently this solution is a gauge one.

Let us find the explicit form of the alternative combination (it should describe a physically observable state)

$$\frac{k_2}{f_2}\varphi_2 - \frac{k_3}{f_3}\varphi_3 = \frac{1}{f}[\begin{vmatrix} \left(k_1^2+k_2^2\right)/(k_0k_2) \\ k_1/k_2 \\ 1 \\ 0 \\ -ik_1k_3^2/(k_0k_2) \\ -ik_3^2/k_0 \\ i\left(k_3k_1^2+k_2^2k_3\right)/k_0k_2 \\ ik_3 \\ -ik_1k_3/k_2 \\ 0 \end{vmatrix} k_2 - \begin{vmatrix} k_3/k_0 \\ 0 \\ 0 \\ 1 \\ ik_1k_3/k_0 \\ ik_2k_3/k_0 \\ -i\left(k_1^2+k_2^2\right)/k_0 \\ -ik_2 \\ ik_1 \\ 0 \end{vmatrix} k_3]e^{-i(k_0x_0+k_jx_j)}, \qquad (50)$$

whence it follows

$$k_2 f\Phi_2 - k_3 f\Phi_3 = \frac{1}{f}\begin{vmatrix} (k_1^2+k_2^2-k_3^2)/k_0 \\ k_1 \\ k_2 \\ -k_3 \\ -2ik_1k_3^2/k_0 \\ -2ik_2k_3^2/k_0 \\ 2i\left(k_1^2+k_2^2\right)k_3/k_0 \\ 2ik_2k_3 \\ -2ik_1k_3 \\ 0 \end{vmatrix} e^{-i(k_0x_0+k_jx_j)} = \Phi_1^{phys}. \qquad (51)$$

It is the mixed solution in the form of the ordinary plane wave, it contains both physical and gauge variables.

Let us collect all four solutions together

$$\Phi_1^{gauge} = \begin{vmatrix} f_0 \\ k_1 \\ k_2 \\ k_3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{vmatrix} e^{-i(f_0x_0+k_jx_j)}, \quad \Phi_2^{gauge} = \frac{1}{f}\begin{vmatrix} k_0 \\ k_1 \\ k_2 \\ k_3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{vmatrix} e^{-i(k_0x_0+k_jx_j)},$$

$$\Phi_1^{phys} = \frac{1}{f}\begin{vmatrix} (k_1^2+k_2^2-k_3^2)/k_0 \\ k_1 \\ k_2 \\ -k_3 \\ -2ik_1k_3^2/k_0 \\ -2ik_2k_3^2/k_0 \\ 2i\left(k_1^2+k_2^2\right)k_3/k_0 \\ 2ik_2k_3 \\ -2ik_1k_3 \\ 0 \end{vmatrix} e^{-i(k_0x_0+k_jx_j)}, \quad \Phi_2^{phys} = \Phi_4 = B_3\begin{vmatrix} -ik_1/(k_0k_2) \\ -i/k_2 \\ 0 \\ 0 \\ -\left(k_2^2+k_3^2\right)/(k_0k_2) \\ k_1/k_0 \\ (k_1k_3)/(k_0k_2) \\ 0 \\ -k_3/k_2 \\ 1 \end{vmatrix} e^{-i(k_0x_0+k_jx_j)}. \quad (52)$$

3.2. The gauge degrees of freedom

Let us turn to additional study of solutions $\Phi_1, \Phi_2, \Phi_3, \Phi_4$. By definition, the gauge solutions should not contribute to the energy-momentum tensor of the Maxwell field.

In order to find this tensor, we need the matrix η of the bilinear form. It should satisfy the following conditions

$$\eta^{-1}(\Gamma^a)^+\eta = -\Gamma^a, \quad \tilde{\Gamma}^a\eta = -\eta\Gamma^a, \quad \eta = \begin{vmatrix} B_{4\times4} & 0 \\ 0 & D_{6\times6} \end{vmatrix}; \quad (53)$$

the matrix of bilinear form were given in section 2.

Expression for tensor $R^a_{\ b}$ was given in section 2

$$R^a_{\ b}(x) = \Phi^+\eta\Gamma^a\partial_b\Phi = -i[H_1^+BK^ak_bH_2 + H_2^+DL^ak_bH_1] = I + II. \quad (54)$$

Further we find explicit form of the tensor $R^a_{\ b}(x)$ for 4 independent solutions:

$$\begin{aligned} &\Phi_1^{gauge}, \quad I \equiv 0, \quad II \equiv 0, \quad R^a_{\ b} \equiv 0, \\ &\Phi_2^{gauge}, \quad I \equiv 0, \quad II \equiv 0, \quad R^a_{\ b} \equiv 0; \end{aligned} \quad (55)$$

consider the physical solutions

$$\Phi_1^{phys}, R^a_{\ b} = 4\left(k_1^2+k_2^2\right)k_3^2\frac{1}{k_0}\frac{1}{f^2}\begin{vmatrix} 2k_0 & 2k_1 & 2k_2 & 2k_3 \\ k_0-k_1 & (k_0-k_1)k_1/k_0 & (k_0-k_1)k_2/k_0 & (k_0-k_1)k_3/k_0 \\ k_0-k_2 & (k_0-k_2)k_1/k_0 & (k_0-k_2)k_2/k_0 & (k_0-k_2)k_3/k_0 \\ k_0-k_3 & (k_0-k_3)k_1/k_0 & (k_0-k_3)k_2/k_0 & (k_0-k_3)k_3/k_0 \end{vmatrix};$$

$$\Phi_2^{phys}, R^a_{\ b} = B_3^2\frac{k_0^2-k_1^2}{k_2^2}\frac{1}{k_0}\begin{vmatrix} 2k_0 & 2k_1 & 2k_2 & 2k_3 \\ k_0-k_1 & (k_0-k_1)k_1/k_0 & (k_0-k_1)k_2/k_0 & (k_0-k_1)k_3/k_0 \\ k_0-k_2 & (k_0-k_2)k_1/k_0 & (k_0-k_2)k_2/k_0 & (k_0-k_2)k_3/k_0 \\ k_0-k_3 & (k_0-k_3)k_1/k_0 & (k_0-k_3)k_2/k_0 & (k_0-k_3)k_3/k_0 \end{vmatrix}.$$

Two last tensors differ from zero, so they relate to physically observable states of the Maxwell field.

3.3. Conclusion

In this section, we have shown that for the Maxwell field, two gauge solutions and two physically observable solutions exist.

So we can make general conclusion that three models by Maxwell, Ogievetsky – Polubarinov – Calb – Ramond, and Stueckelberg, substantially differ from each other. They describe different physical massless fields.

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