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ON A FINITE GROUP FACTORIZED BY CONDITIONALLY SEMINORMAL SUBGROUPS*

The subgroups A and B are said to be cc*-permutable, if A permutes with B^x for some $x \in \langle A, B\rangle$. A subgroup A of a finite group G is called conditionally seminormal in G, if there exists a subgroup T of G such that $G = AT$ and A is* cc*-permutable with all subgroups of T. The groups $G = G_1G_2 \ldots G_n$ with pairwise permutable subgroups $G_1, \ldots, G_n$ such that G_i and G_j are conditionally seminormal in G_iG_j for any $i, j \in \{1, \ldots, n\}$, $i \neq j$, are studied.*

Key words: *finite factorized group, conditionally seminormal subgroups.*

О конечной группе, факторизуемой условно полунормальными подгруппами

Подгруппы A и B называются cc*-перестановочными, если A перестановочна с B^x для некоторого $x \in \langle A, B\rangle$. Подгруппа A конечной группы G называется условно полунормальной G, если существует подгруппа T группы G такая, что $G = AT$ и A* cc*-перестановочна с каждой подгруппой из T. В работе изучены группы $G = G_1G_2 \ldots G_n$ с попарно перестановочными подгруппами $G_1, \ldots, G_n$ такими, что G_i и G_j условно полунормальны G_iG_j для любых $i, j \in \{1, \ldots, n\}$, $i \neq j$.*

Ключевые слова: *конечная факторизуемая группа, условно полунормальные подгруппы.*

Introduction

Throughout this paper, all groups are finite and G always denotes a finite group. We use the standard notations and terminology of [1; 2].

It is well known that AB is a subgroup of G if and only if $AB = BA$, that is, if the subgroups A and B permute. Should it happen that AB coincides with the group G, then G is said to be *factorized* by its subgroups A and B. A group G is said to be the *product of its pairwise permutable subgroups* $G_1, G_2, \ldots, G_n$ if $G = G_1G_2 \ldots G_n$ and $G_iG_j = G_jG_i$ for all integers i and j with $i, j \in \{1, 2, \ldots, n\}$. In many of the most relevant results in group theory structure properties of a product of pairwise permutable subgroups are deduced from properties of the factors. For example, a group is soluble if and only if it is the product of pairwise permutable Sylow subgroups; if a group is the product of pairwise permutable nilpotent subgroups, then it is soluble; a group which is the product of pairwise permutable cyclic subgroups is supersoluble.

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We say that two subgroups A and B of G are *mutually permutable* [3] if A permutes with every subgroup of B and B permutes with every subgroup of A. The following definition give a very special case of a product with pairwise permutable subgroups $G_1, G_2, \ldots, G_n$. A group $G = G_1 G_2 \ldots G_n$ is said to be the *pairwise mutually permutable product* of the subgroups $G_1, G_2, \ldots, G_n$ if G_i and G_j are mutually permutable subgroups for all $i, j \in \{1, 2, \ldots, n\}$. The monograph [1] contains detailed information on the structure of groups factorized by finitely many pairwise mutually permutable subgroups.

In this direction, the results given in the following theorem are well known.

Theorem A. *Let $G = G_1 G_2 \ldots G_n$ be the product of the pairwise mutually permutable subgroups $G_1, \ldots, G_n$. Let $\mathfrak{F}$ be a saturated formation such that $\mathfrak{U} \subseteq \mathfrak{F}$. Then:*

1) if G_i belongs to $\mathfrak{F}$ for all $i \in \{1 \ldots, n\}$ and G' is nilpotent, then G belongs to $\mathfrak{F}$, see [1, Theorem 5.2.21]*;*

2) if G_1 is supersoluble and G_i is nilpotent for every $i \in \{2, \ldots, n\}$, then G is supersoluble, see [1, Theorem 4.1.35].

Recall that a subgroup T of a group G is said to be a *supplement* to a subgroup A in G if $G = AT$. A subgroup A of a group G is called *seminormal* in G, if there exists a supplement B such that $G = AB$ and AX is a subgroup of G for every subgroup X of B. A weaker condition for permutability was given in [4]: the subgroups A and B are said to be cc*-permutable*, if A permutes with B^x for some $x \in \langle A, B \rangle$. It is quite natural to study the cc-permutability between subgroups of A and subgroups of supplement to A. Using this approach, the concept of conditionally seminormal subgroups was introduced in [5].

Definition. A subgroup A of a finite group G is called *conditionally seminormal* in G, if there exists a subgroup T of G such that $G = AT$ and A is cc-permutable with all subgroups of T.

In this case, we say that T is a *conditionally supplement* to A in G. If $G = AB$ is a product of mutually cc-permutable subgroups A and B, then A and B are conditionally seminormal subgroups of G. The converse is false. An example is the dihedral group $G = \langle a \rangle \rtimes \langle c \rangle \simeq D_{24}$, $|a| = 12$, $|c| = 2$, considered above. In [5] the structure of a group with given systems of conditionally seminormal subgroups was established.

In present article, we continue our research in this direction and extend the above-mentioned results of Theorem A. The groups $G = G_1 G_2 \ldots G_n$ with pairwise permutable subgroups $G_1, \ldots, G_n$ such that G_i and G_j are conditionally seminormal in $G_i G_j$ for any $i, j \in \{1, \ldots, n\}$, $i \neq j$ are studied.

We prove the following

Theorem B. *Let $G = G_1 G_2 \ldots G_n$ be the product of the pairwise permutable subgroup $G_1, \ldots, G_n$ such that G_i and G_j are conditionally seminormal in $G_i G_j$ for any $i, j \in \{1, \ldots, n\}$, $i \neq j$. Let $\mathfrak{F}$ be a saturated formation such that $\mathfrak{U} \subseteq \mathfrak{F}$.*

1. *Suppose that G_i belongs to $\mathfrak{F}$ for all $i \in \{1 \ldots, n\}$. Then $G^{\mathfrak{F}} \leq (G')^{\mathfrak{N}}$.*

2. *Suppose that G_i belongs to $\mathfrak{F}$ for all $i \in \{1 \ldots, n\}$, $\mathfrak{F} \subseteq \mathcal{D}$ and $(|G_i|, |G_j|) = 1$ for any $i, j \in \{1, \ldots, n\}$, $i \neq j$. Then $G \in \mathfrak{F}$.*

3. *If G_1 is supersoluble and G_i is nilpotent for every $i \in \{2, \ldots, n\}$, then G is supersoluble.*

That the condition of being nilpotent in Theorem B (3) cannot be weakened to being supersoluble for at least one of the factors G_i, $i \in \{2,\dots,n\}$, is demonstrated by the following.

Example. A non-supersoluble group $T = V \rtimes Q$, where $V =\langle a,b \rangle \simeq E_{5^2}$ and $Q =\langle x, y \rangle \simeq Q_8$, was constructed in [6]. The group T is representable as a product of two normal supersoluble subgroups $A = V\langle x \rangle$ and $B = V\langle y \rangle$. Consider a group $G = T \times C$, where $C \simeq Z_p$. Obviously, the group $G = ABC$ is non-supersoluble and is a product of subgroups A, B, and C normal in G.

Preliminary results

In this section, we give some definitions and basic results which are essential in the sequel.

If $H \le G$ and $H \ne G$, we write $H < G$. The notation $H \trianglelefteq G$ means that H is a normal subgroup of a group G. Denote by G', $Z(G)$, $F(G)$ and $\Phi(G)$ the derived subgroup, centre, Fitting and Frattini subgroups of G respectively, and by $O_p(G)$ and $O_{p'}(G)$ the greatest normal p- and p'-subgroups of G respectively. Denote by $\pi(G)$ the set of all prime divisors of order of G. The semidirect product of a normal subgroup A and a subgroup B is written as follows: $A \rtimes B$.

A group whose chief factors have prime orders is called *supersoluble*. Recall that a p*-closed* group is a group with a normal Sylow p-subgroup and a p*-nilpotent* group is a group with a normal Hall p'-subgroup.

The monograph [7] contains the necessary information of the theory of formations. The formations of all nilpotent, p-groups, supersoluble groups and all groups which have an ordered Sylow tower of supersoluble type are denoted by $\mathfrak{N}$, $\mathfrak{N}_p$, $\mathfrak{U}$ and $\mathcal{D}$, respectively. A formation $\mathfrak{F}$ is said to be *saturated* if $G/\Phi(G) \in F$ implies $G \in \mathfrak{F}$. Let $\mathbb{P}$ be the set of all prime numbers. A *formation function* is a function f defined on $\mathbb{P}$ such that $f(p)$ is a, possibly empty, formation. A formation $\mathfrak{F}$ is said to be *local* if there exists a formation function f such that $G \in \mathfrak{F}$ if and only if for any chief factor H/K of G and any $p \in \pi(H/K)$, one has $G/C_G(H/K) \in f(p)$. We write $\mathfrak{F} = LF(f)$ and f is a local definition of $\mathfrak{F}$. A chief factor H/K of a group G in this case is called f-central. It is clear that $\mathfrak{F} = LF(f) = \{G \mid G/F_p(G) \in f(p)\}$. Here $F_p(G)$ is the greatest normal p-nilpotent subgroup of G. By [7, Theorem IV.3.7], among all possible local definitions of a local formation $\mathfrak{F}$ there exists a unique f such that f is integrated (i.e. $f(p) \subseteq \mathfrak{F}$ for all $p \in \mathbb{P}$) and full (i. e. $f(p) = \mathfrak{N}_p f(p)$ for all $p \in \mathbb{P}$). Here $\mathfrak{N}_p$ is the formation of all p-groups. Such local definition f is said to be *canonical local definition* of $\mathfrak{F}$. By [7, Theorem IV.4.6], a formation is saturated if and only if it is local.

If a group G contains a maximal subgroup M with trivial core, then G is said to be *primitive* and M is its *primitivator*.

Lemma 1. *Let $\mathfrak{F}$ be a saturated formation and G be a group. Assume that $G \notin \mathfrak{F}$, but $G/N \in \mathfrak{F}$ for all non-trivial normal subgroups N of G. Then G is a primitive group.*

Lemma 2. *([2, Theorem II.3.2]) Let G be a soluble primitive group and M is a primitivator of G. Then the following statements hold:*

(1) $\Phi(G) = 1$;

(2) $F(G) = C_G(F(G)) = O_p(G)$ *and* $F(G)$ *is an elementary abelian subgroup of order* p^n *for some prime* p *and some positive integer* n*;*

(3) G *contains a unique minimal normal subgroup* N *and moreover,* $N = F(G)$*;*

(4) $G = F(G) \rtimes M$ *and* $O_p(M) = 1$.

Lemma 3 ([8, Lemma 6]). *Let* G *be a soluble group. Assume that* $G \notin \mathfrak{U}$*, but* $G/K \in \mathfrak{U}$ *for every non-trivial normal subgroup* K *of* G*. Then the following hold:*

(1) G *contains a unique minimal normal subgroup* N*,* $N = F(G) = \mathrm{O}_p(G) = C_G(N)$ *for some* $p \in \pi(G)$*;*

(2) $Z(G) = \mathrm{O}_{p'}(G) = \Phi(G) = 1$*;*

(3) G *is primitive;* $G = N \rtimes M$*, where* M *is maximal in* G *with trivial core;*

(4) N *is an elementary abelian subgroup of order* p^n*,* $n > 1$*;*

(5) if M *is abelian, then* M *is a cyclic group of order dividing* $p^n - 1$ *and* n *is the smallest positive integer such that* $p^n \equiv 1 (\mathrm{mod}\, |M|)$.

Lemma 4 ([9, Theorem 2]). *Suppose* G *is the product of pairwise permutable subgroups* $G_1, \ldots, G_n$*,* $n \geq 2$*, such that* G_iG_j *is a soluble subgroup for any* $i, j \in \{1, \ldots, n\}$*,* $i \neq j$*. Then* G *is soluble.*

Lemma 5. *Let* $G = G_1G_2 \ldots G_n$ *be a group with pairwise permutable subgroups* $G_1, \ldots, G_n$ *and* $H \leq G_1 \cap \ldots \cap G_n$*. If* H *is subnormal in* G_i *for every* $i \in \{1, \ldots, n\}$*, then* H *is subnormal in* G*.*

Proof. We argue by induction of n. By [10, Theorem 1], the result is clear when $n = 2$. Let $K = G_1G_2 \ldots G_{n-1}$. By induction, H is subnormal in K. Then by [10, Theorem 1], H is subnormal in $G = KG_n$.

Lemma 6. *Let* G *be soluble. If* G *has a subgroup* H *of prime index, then* G/H_G *is supersoluble.*

Proof. Suppose that H is not normal in G. Then $H_G \neq H$ and G/H_G is primitive with stabilizer H/H_G. By [7, Theorem 15.6],

$$G/H_G = (P/H_G) \rtimes (H/H_G), \; P/H_G = C_{G/H_G}(P/H_G).$$

Let $|G:H| = p$, where p is prime. Then

$$|G/H_G : H/H_G| = |G:H| = p, \; |P/H_G| = p.$$

The subgroup H/H_G is cyclic, as the automorphism group of P/H_G of prime order. Hence G/H_G is supersoluble. If H is normal in G, then $H = H_G$ and G/H_G is supersoluble. Lemma is proved.

Lemma 7. *Let* A *be a conditionally seminormal subgroup of* G *and* Y *be a conditionally supplement to* A *in* G*. Then the following statements hold:*

(1) A *is a conditionally seminormal subgroup of* H *for any subgroup* H *of* G *such that* $A \leq H$*;*

(2) AN/N *is a conditionally seminormal subgroup of* G/N *for any* $N \trianglelefteq G$*;*

(3) for any $X \leqslant Y$ there exists $y \in Y$ such that $AX^{y} \leqslant G$. In particular, $AM \leqslant G$ for some maximal subgroup M of Y and $AH \leqslant G$ for some Hall π-subgroup of H of soluble Y and any $\pi \subseteq \pi(G)$;

(4) $AK \leqslant G$ for every subnormal subgroup K of Y;

(5) $AK^{g} \leqslant G$ for any $g \in G$ and any subnormal subgroup K of Y;

(6) A^{x} is a conditionally seminormal subgroup of G and Y^{x} is a conditionally supplement to A^{x} in G for any $x \in G$;

(7) if A is a maximal subgroup of G, then $|G:A|$ is prime;

(8) if Y is soluble and A is r-closed, then Sylow r-subgroup A_r of A is subnormal in G, where r is the greatest prime in $\pi(G)$;

(9) if A is 2-nilpotent, then the derived subgroup A' is subnormal in G.

Proof. The validity of statements 1–8 directly follows from [5, Lemma 6]. We prove statement (9). We proceed by induction on $|G|$. By (3), for every $p \in \pi(Y)$ there exists a Sylow p-subgroup Y_p of Y such that $AY_p \leq G$. Suppose that $AY_p < G$ for every $p \in \pi(Y)$. Then by (1), A is a conditionally seminormal subgroup in AY_p and by induction, A' is subnormal in AY_p. It is clear that for every $p \in \pi(G)$ there exists a Sylow p-subgroup R of G such that $R \leq AY_p$. Since $A' \leq \langle A', R\rangle \leq AY_p$, we have A' is subnormal in $\langle A', R\rangle$. By [11, Theorem], A' is subnormal in G.

Hence we consider that $G = AY_q$ for some $q \in \pi(Y)$. By (4), A is seminormal in G. Since A is 2-nilpotent, A^G is soluble by [12, Lemma 10]. Hence $G = AY_q = A^G Y_q$ is soluble. By (4), Y_q is a minimal conditionally supplement to A in G and $AT < G$ for some maximal subgroup T of Y_q. Because A is a conditionally seminormal subgroup in AT, we have by induction, A' is subnormal in AT. Since $|G:AT| = q$, it follows that by Lemma 6, $G/(AT)_G$ is supersoluble and hence the derived subgroup

$$(G/(AT)_G)' = G'(AT)_G/(AT)_G$$

is nilpotent. Since $A' \leq G'$, we have

$$A'(AT)_G/(AT)_G \leq G'(AT)_G/(AT)_G$$

and hence $A'(AT)_G$ is subnormal in G. It is clear that $A' \leq A'(AT)_G \leq AT$. Since A' is subnormal in AT, A' is subnormal in $A'(AT)_G$ and A' is subnormal in G.

Lemma 8. 1. *Suppose that A and B are conditionally seminormal subgroups of a group G and $G = AB$.*

(1. 1) If A and B have Sylow towers of supersoluble type then G has a Sylow tower of supersoluble type.

(1.2) If A is nilpotent and B is supersoluble then G is supersoluble.

2. Let $G = G_1G_2\ldots G_n$ be a group with pairwise permutable subgroups $G_1,\ldots,G_n$ such that G_i and G_j are conditionally seminormal in G_iG_j for any $i, j \in \{1,\ldots,n\}$, $i \neq j$.

(2.1) If a Sylow p-subgroup P of G is normal in G and is abelian, then $P \cap G_i$ is normal in G for every i.

(2.2) If $G_1,\dots,G_n$ have an ordered Sylow tower of supersoluble type, then G has an ordered Sylow tower of supersoluble type.

Proof. 1.1. Assume that the claim is false and let G be a minimal counterexample. Let N be a non-trivial normal subgroup of G. The subgroups $AN/N \simeq A/A\cap N$ and $BN/N \simeq B/B\cap N$ is conditionally seminormal in $G/R=(AR/R)(BR/R)$ by Lemma 7 (2) and have an ordered Sylow tower of supersoluble type. By induction, G/N has an ordered Sylow tower of supersoluble type.

We show that G is soluble. By Lemma 7 (9), $A^{'}$ and $B^{'}$ are subnormal in G. If A and B are abelian, then by Theorem Itô, G is soluble. Hence we consider that either $A^{'}\neq 1$ or $B^{'}\neq 1$. Suppose that $A^{'}\neq 1$. Since A has Sylow towers of supersoluble type, $(A^{'})^G$ is suluble. If $(A^{'})^G=G$, then G is soluble. If $(A^{'})^G<G$, then $G/(A^{'})^G$ has Sylow towers of supersoluble type. Hence G is soluble.

Let $r\in\pi(G)$ and r be the greatest. It is clear that a Sylow r-subgroup A_r is normal in A. Then by Lemma 7 (8), A_r is subnormal in G. Similarly, a Sylow r-subgroup B_r of B is subnormal in G. Since $R=A_rB_r$ is a Sylow subgroup of G, we have G is r-closed. Since G/R has an ordered Sylow tower of supersoluble type by induction, it follows that G has a Sylow tower of supersoluble type.

1.2. Assume that the claim is false and let G be a minimal counterexample. If N is a non-trivial normal subgroup of G, then the subgroups AN/N and BN/N are conditionally seminormal in G/N by Lemma 7 (2), $AN/N\simeq A/A\cap N$ is nilpotent and $BN/N\simeq B/B\cap N$ is supersoluble. Then by induction, $G/N=(AN/N)(BN/N)$ is supersoluble. By (1.1), G has an ordered Sylow tower of supersoluble type. By Lemma 3, $\Phi(G)=1$, $G=N\rtimes M$, where $N=C_G(N)=F(G)=\mathrm{O}_p(G)$ is a unique minimal normal subgroup of G. Besides, $N=G_p$ is the Sylow p-subgroup for the greatest $p\in\pi(G)$. Since $G=AB$, it follows that $N=A_pB_p$, where A_p and B_p are Sylow p-subgroups of A and B respectively.

Let Y be conditionally supplement to A in G. Suppose that $A_p=1$. Then $N=B_p\le B$. We choose a minimal normal subgroup N_1 of B such that $N_1\le N$. Since B is supersoluble, we have $|N_1|=p$. Since $N_1\le N\le Y$, it follows that there exists a subgroup $AN_1=N_1\rtimes A$ and N_1 is normal in G, a contradiction. Thus the assumption $A_p=1$ is false and $A_p\neq 1$.

Assume that $B_p=1$. Hence $N=A_p\le A$ and $N=A$. Then $B\cap N=1$ and B is maximal in G. Since B is conditionally seminormal in G, we have by Lemma 7 (7), the index of B in G is prime, a contradiction.

By Lemma 7 (3), $AY_1\leqslant G$ for some Hall $p^{'}$-subgroup Y_1 of Y. It is clear that $Y_1\le N_G(A_p)$, since G is p-closed. Since N is abelian, a Sylow p-subgroup Y_p of Y centralizes A_p and A_p is normal in G, hence $A_p=N$. Because A is nilpotent and by Lemma 3 (1), it follows that $A=N$. Since B is supersoluble, we have B_p is normal in B. In this case, B_p is normal in $N=A$ and therefore is normal in G. Thus $B_p=N$ and $G=AB=NB=B$ is supersoluble, a contradiction.

2.1. We consider the following representation

$$G=(G_1G_i)\dots G_i\dots(G_nG_i).$$

Since G_i is conditionally seminormal in G_iG_j, there exists a subgroup H_j such that $G_iH_j = G_iG_j$. Let $(H_j)_p$ is a Sylow p-subgroup of H_j. By Lemma 7 (3), H_j has a Hall p'-subgroup $(H_j)_{p'}$ of H_j such that $G_i(H_j)_{p'}$ is a subgroup of G. It's obvious that $P \cap G_i$ is a Sylow p-subgroup of G_i. Denote by $(G_i)_p = P \cap G_i$. Hence $P \cap G_i(H_j)_{p'} = (G_i)_p$ and $(G_i)_p$ is normal in $G_i(H_j)_{p'}$. Therefore $(H_j)_{p'} \leq N_G((G_i)_p)$. Since $(H_j)_p$ and $(G_i)_p$ are contained in abelian subgroup P, we have $(H_j)_p \leq C_G((G_i)_p)$. So $(G_i)_p$ is normal in

$$G = (G_1G_i)\dots G_i \dots (G_nG_i) = (H_1G_i)\dots G_i \dots (H_nG_i) =$$
$$= ((H_1)_p(H_1)_{p'}G_i)\dots G_i \dots ((H_n)_p(H_n)_{p'}G_i).$$

2.2. If $n = 2$, then Lemma is true by (1.1). Hence $n \geq 3$. By induction, every product G_iG_j has an ordered Sylow tower of supersoluble type. By Lemma 4, G is soluble. Let $r \in \pi(G)$ and r be the greatest. It is clear that a Sylow r-subgroup $(G_1)_r$ is normal in G_1. Then by Lemma 7 (8), $(G_1)_r$ is subnormal in G_1G_i for any $i \neq 1$. By Lemma 5, $(G_1)_r$ is subnormal in $G = G_1(G_1G_2)\dots(G_1G_n)$. Hence $(G_i)_r$ is subnormal in G for every $i \in \{1,\dots,n\}$. Since $R = (G_1)_r \dots (G_n)_r$ is a Sylow subgroup of G, it follows that G is r-closed. The subgroups $G_iR/R \simeq G_i/G_i \cap R$ and $G_jR/R \simeq G_j/G_j \cap R$ are conditionally seminormal in G_iG_jR/R by Lemma 7 (3) and have an ordered Sylow tower of supersoluble type. By induction, G/R has an ordered Sylow tower of supersoluble type, hence G has an ordered Sylow tower of supersoluble type.

Proof of Theorem B

1. We consider the case when the derived subgroup G' is nilpotent. Then G is soluble. Assume that $G \notin \mathfrak{F}$. Let N be a non-trivial normal subgroup of G. The quotients

$$G/N = \prod_i (G_iN/N), G_iN/N \simeq G_i/G_i \cap N.$$

Hence the subgroups $G_iN/N \in \mathfrak{F}$ for all i and by Lemma 7 (3), G_iN/N and G_jN/N are conditionally seminormal in G_iG_jN/N for any $i \neq j$.

Since

$$(G/N)' = G'N/N \simeq G'/G' \cap N,$$

it follows that the derived subgroup $(G/N)'$ is nilpotent. Thus the hypotheses of the theorem hold for G/N. By induction, $G/N \in \mathfrak{F}$. Since $\mathfrak{F}$ is saturated, we have that G is primitive by Lemma 1. Hence $\Phi(G) = 1$, $G = N \rtimes M$ and $N = C_G(N) = F(G) = \mathrm{O}_p(G)$ is a unique minimal normal subgroup of G by Lemma 2. Because G' is nilpotent, we have $N = G'$ and G/N is abelian. Let P be a Sylow p-subgroup of G. Then $PN/N = P/N$ is a Sylow p-subgroup of G/N. Hence P is normal in G, $N = P$ and $M = G_{p'}$ is abelian.

Without loss of generality, we assume that p divides the order of G_1. By Lemma 8 (2.1), $N \leq G_1$. Since G_1 is normal in G, it follows that $N = F(G_1) = F_p(G_1)$. Since $\mathfrak{F}$ is saturated, there exists the canonical local definition f. Hence $\mathfrak{F} = LF(f)$, $f(p) \subseteq \mathfrak{F}$ and

$f(p) = \mathfrak{N}_p f(p)$. By hypothesis, $G_1 \in \mathfrak{F}$. Then by definition of formation function, $G_1 / N = G / F_p(G_1) \in f(p)$.

Let $K = G_2 G_3 \ldots G_n$ and p divides the order of K. Hence p divides the order at least one of the subgroups G_i, $i \in \{2 \ldots, n\}$. By Lemma 8 (2.1), $N \leq K$. By induction, $K \in \mathfrak{F}$. Then, proving as above, $K / N \in f(p)$.

Suppose p does not divide the order of K. Since for every $i \in \{2 \ldots, n\}$ G_i is conditionally seminormal in $G_i G_1$, it follows that there exists a subgroup T such that $G_i T = G_i G_1$ and $G_i X$ is a subgroup of G for every subnormal subgroup X of T by Lemma 7 (4). Because $N \leq T$ and G is p-closed, we have $G_i \leq N_G(U)$ for every subgroup U of N. By [13, Lemma 2.6], K is cyclic of order dividing $p-1$. Then $K \in g(p)$, where g is a canonical local definition of a saturated formation $\mathfrak{U}$. Since $\mathfrak{U} \subseteq \mathfrak{F}$, we have by [7, Proposition IV.3.11], $g(p) \subseteq f(p)$. Hence $K \in f(p)$.

Thus G / N is the product of normal subgroups G_1 / N and KN / N such that each of them belongs to $f(p)$. Since G / N is Abelian, $[G_1 N / N, KN / N] = 1$ and $G / N \in f(p)$ by [13, Lemma 2.4]. Because $N \in \mathfrak{N}_p$, we have $G \in \mathfrak{N}_p f(p) = f(p) \subseteq \mathfrak{F}$. Hence the assumption is wrong.

Let $(G')^{\mathfrak{N}} \neq 1$. We show that the quotient $G / (G')^{\mathfrak{N}}$ belongs to F. Since

$$(G / (G')^{\mathfrak{N}})' = G' (G')^{\mathfrak{N}} / (G')^{\mathfrak{N}} = G' / (G')^{\mathfrak{N}},$$

we have the $(G / (G')^{\mathfrak{N}})'$ is nilpotent. The quotients

$$G / (G')^{\mathfrak{N}} = \prod_i (G_i (G')^{\mathfrak{N}} / (G')^{\mathfrak{N}})),$$

$$G_i (G')^{\mathfrak{N}} / (G')^{\mathfrak{N}} \simeq G_i / G_i \cap (G')^{\mathfrak{N}},$$

hence the subgroups $G_i (G')^{\mathfrak{N}} / (G')^{\mathfrak{N}}$ for any $i \in \{1 \ldots, n\}$ belong to $\mathfrak{F}$ and by Lemma 7 (3), $G_i (G')^{\mathfrak{N}} / (G')^{\mathfrak{N}}$ and $G_j (G')^{\mathfrak{N}} / (G')^{\mathfrak{N}}$ are conditionally seminormal in $G_i G_j (G')^{\mathfrak{N}} / (G')^{\mathfrak{N}}$. Arguing as above, we see that $G / (G')^{\mathfrak{N}}$ belongs to $\mathfrak{F}$.

2. Assume that the claim is false and let G be a minimal counterexample. Let N be a non-trivial normal subgroup of G. Since

$$(|G_i N / N|, |G_j N / N|) = (|G_i / G_i \cap N|, |G_j / G_j \cap N|) = 1$$

for any $i \neq j$, it follows that G / N satisfies all the requirements of the theorem and by induction, $G / N \in \mathfrak{F}$.

Let p be the greatest prime number of $\pi(G)$. Since $(|G_i|, |G_j|) = 1$ for any $i \neq j$, without loss of generality we have that $p \in \pi(G_1)$. Let Y be a conditionally supplement to G_1 in G. By Lemma 7 (3), for every $q \in \pi(Y)$ there exists a Sylow q-subgroup Y_q of Y such that $G_1 Y_q \leq G$. Since $G_1 \in \mathfrak{F} \subseteq \mathcal{D}$, G_1 is p-closed. If $q \neq p$, then by Lemma 7 (8), $(G_1)_p$ is subnormal in $G_1 Y_q$ and consequently $(G_1)_p$ is normal in $G_1 Y_q$ and $Y_q \leq N_G((G_1)_p)$. If $q = p$, then $Y_p \leq N_G((G_1)_p)$. So, $(G_1)_p$ is normal in $G = G_1 Y$, since $Y_q \leq N_G((G_1)_p)$ for any $q \in \pi(Y)$.

It is clear that $(G_1)_p$ is the Sylow p-subgroup of G. Let $P=(G_1)_p$. Then G is soluble, because $G/P\in\mathfrak{F}\subseteq\mathcal{D}$.

By Lemmas 1 and 2, G is primitive and $G=N\rtimes M$, where

$$N=C_G(N)=F(G)=\mathrm{O}_p(G)=P-$$

is a unique minimal normal subgroup of G and M is a Hall p'-subgroup of G. It can be assumed that $M=(G_1)_{p'}\dots(G_n)_{p'}$ for some Hall p'-subgroups $(G_1)_{p'},\dots,(G_n)_{p'}$ of $G_1,\dots,G_n$, respectively. Let $K=G_2G_3\dots G_n$. Since $(|G_1|,|K|)=1$ and $P\le G_1$, it follows that K is a p'-subgroup and $M=(G_1)_{p'}K$.

Since G_j is conditionally seminormal in G, $j>1$, we have that $G=G_jH_j$, where H_j is a conditionally supplement to G_j in G and $P\le H_j$. Let X be an arbitrary subgroup of P. By Lemma 7 (4), $G_jX\le G$. Since G is p-closed, $G_j\le N_G(X)$ for any $j>1$ and $K\le N_G(X)$. By [13, Lemma 2.6] K is a cyclic group of order dividing $p-1$ and $[(G_1)_{p'},K]=1$.

Since K is a cyclic group of order dividing $p-1$. Hence $K\in g(p)$, where g is a canonical local definition of a saturated formation $\mathfrak{U}$. Since $\mathfrak{U}\subseteq\mathfrak{F}$, we have by [7, Proposition IV.3.11], $g(p)\subseteq f(p)$, where f is a canonical local definition of a saturated formation $\mathfrak{F}$. Hence $K\in f(p)$. Because $G_1\in\mathfrak{F}$ and $F_p(G_1)=P$, we have that $G_1/F_p(G_1)=G_1/P\cong(G_1)_{p'}\in f(p)$. Since $[(G_1)_{p'},K]=1$, $(G_1)_{p'}\in f(p)$, $K\in f(p)$ and $f(p)$ is a formation, it follows that by [13, Lemma 2.4], $G/P\cong M=(G_1)_{p'}K\in f(p)$. Because $P\in\mathfrak{N}_p$, we have $G\in\mathfrak{F}$, a contradiction.

3. If $n=2$, then Theorem is true by Lemma 8 (1.2). Hence $n\ge 3$. By Lemma 8 (2.2), G has an ordered Sylow tower of supersoluble type.

Assume that the claim is false and let G be a minimal counterexample. Let N be a non-trivial normal subgroup of G. The subgroups G_iN/N and G_jR/R are conditionally seminormal in G_iG_jR/R by Lemma 7 (3), $G_iN/N\simeq G_i/G_i\cap N$ is nilpotent for all $i\in\{2,\dots,n\}$ and $G_1N/N\simeq G_1/G_1\cap N$ is supersoluble. By induction, G/N is supersoluble. By Lemma 3, G is primitive, G contains a unique minimal normal subgroup N such that $N=C_G(N)=\mathrm{O}_p(G)=F(G)$ for some $p\in\pi(G)$ and N is an elementary abelian subgroup of order p^n, where $n>1$. Since G has an ordered Sylow tower of supersoluble type, we have $N=G_p$, where p is the greatest in $\pi(G)$.

Suppose that the order of G_i is divisible by p for some i. By Lemma 8 (2.1), $N\le G_i$. If $i>1$, then G_i is nilpotent and hence $G_i=N$. Thus G_i either coincides with N, or is p'-subgroup for each $i\in\{2,\dots,n\}$.

If $G_2G_3\dots G_n$ is a p'-subgroup, then $N\le G_1$. Let N_1 be a minimal normal subgroup of G_1 such that $N_1\le N$. Then $|N_1|=p$, because G_1 is supersoluble. By hypothesis, G_i is conditionally seminormal in G_iG_1. Hence there exists a subgroup R such that $G_iR=G_iG_1$. Since G_i is a p'-subgroup, $N\le R$ and G_iN_1 is a subgroup of G by Lemma 7 (4). Because G is p-closed, we have $G_i\le N_G(N_1)$. Therefore N_1 is normal in G, a contradiction. Thus

$G_2G_3\dots G_n$ is not a p'-subgroup and G has the representation $G = G_1N(G_{s_1}\dots G_{s_r})$, where G_{s_j} is a nilpotent p'-subgroup of G and N is coincides with all nilpotent subgroups G_i, whose order is divisible by p.

Since G_{s_j} is conditionally seminormal in $G_{s_j}N$, it follows that there exists a subgroup T such that $G_{s_j}T = G_{s_j}N$ and $G_{s_j}X$ is a subgroup of G for every subnormal subgroup X of T by Lemma 7 (4). Because $N \le T$, we have $G_{s_j} \le N_G(U)$ for every subgroup U of N. Since G_1N is supersoluble by Lemma 8 (1.2), G_1N has a normal subgroup V of order p. Now V is normal in G, a contradiction. The theorem is proved.

Conclusions

In present paper we studied the groups $G = G_1G_2\dots G_n$ with pairwise permutable subgroups $G_1,\dots,G_n$ such that G_i and G_j are conditionally seminormal in G_iG_j for any $i, j \in \{1,\dots,n\}$, $i \ne j$. In particular, we obtained a number of conditions for such groups to belong to a saturated formation containing formations of supersoluble groups.

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