УДК 539.12 DOI 10.63874/2218-0303-2026-1-11-20

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SPIN 2 PARTICLE IN THE COULOMB FIELD, STATES WITH MINIMAL VALUE OF THE QUANTUM NUMBER $j = 0$

The main goal of this study is to derive the non-relativistic system of equations for a spin 2 particle in presence of the external Coulomb field, then to solve it and to find the relevant energy spectra. We started with the known radial system of 39 equations derived for the case of a free particle and modified to take into account the presence of the Coulomb field. After eliminating the 28 components related to vector and 3-rank tensor, we get the system of second order for 11 components referring to scalar and symmetric tensor. Previously, the problem of a non-relativistic spin 2 particle in external Coulomb field was studied. In turned out that this study did not cover the case of minimal value of the quantum number, $j = 0$. In the present paper, this special situation is solved, the corresponding energy spectrum is found.

Key words: *spin 2 particle, angular momentum, coulomb field, non-relativistic equation, projective operators, energy spectrum.*

Частица со спином 2 в кулоновском поле, состояния с минимальным значением квантового числа $j = 0$

Основная цель исследования – вывести нерелятивистскую систему уравнений для частицы со спином 2 в присутствии внешнего кулоновского поля, затем решить ее и найти соответствующие энергетические спектры. В работе использована известная радиальная система из 39 уравнений, выведенная для случая свободной частицы и модифицированная с учетом присутствия кулоновского поля. После исключения 28 компонент, связанных с вектором и тензором третьего ранга, мы получаем систему второго порядка для 11 компонент, относящихся к скаляру и симметричному тензору. Ранее была изучена задача о нерелятивистской частице со спином 2 во внешнем кулоновском поле. Однако данное исследование не охватывало случай минимального значения квантового числа $j = 0$. В работе исследован именно этот случай, найден соответствующий энергетический спектр.

Ключевые слова: *частица со спином 2, angular momentum, кулоновское поле, нерелятивистское уравнение, проективные операторы, спектр энергии.*

1. Solving the problem

In [1], the problem of a non-relativistic spin 2 particle in external Coulomb field was studied. In turned out that this study did not cover the case of minimal value of the quantum number, $j = 0$. In the present paper, this special situation will be solved. Let us recall only the needed points. The wave function of a relativistic spin 2 particle consists of a scalar (H), 4-vector (H_1), symmetric tensor(H_2), and 3-rank tensor antisymmetric in two indices (H_3); in total, the wave function includes 39 components. On solutions with spherical symmetry the following operators are diagonalized

$$\begin{aligned} &\vec{J}^{\,2} H = j(j+1)H, \quad J_3 H = mH; \qquad \vec{J}^{\,2}_{(1)} H_1 = j(j+1)H_1, \quad J_3^{(1)} H_1 = mH_1; \\ &\vec{J}^{\,2}_{(2)} H_2 = j(j+1)H_2, \quad J_3^{(2)} H_2 = mH_2; \qquad \vec{J}^{\,2}_{(3)} H_3 = j(j+1)H_3, \quad J_3^{(3)} H_2 = mH_3. \end{aligned} \tag{1}$$

To separate the variables, we applied the Wigner D-functions technique $D^{j}_{-m,-s_3}(\phi,\theta,0) = D_{-s_3}$; see in [2; 3], at this the known recurrent relations [2] were used:

$$
\begin{gathered}
\partial_\theta D_0 = +\frac{1}{2}a\,D_{-1} - \frac{1}{2}a\,D_{+1}, \quad \frac{-m}{\sin\theta}D_0 = -\frac{1}{2}a\,D_{-1} - \frac{1}{2}a\,D_{+1}, \\
\partial_\theta D_{+1} = +\frac{1}{2}a\,D_0 - \frac{1}{2}b\,D_{+2}, \quad \frac{-m-\cos\theta}{\sin\theta}D_{+1} = -\frac{1}{2}a\,D_0 - \frac{1}{2}b\,D_{+2}, \\
\partial_\theta D_{-1} = +\frac{1}{2}b\,D_{-2} - \frac{1}{2}a\,D_0, \quad \frac{-m+\cos\theta}{\sin\theta}D_{-1} = -\frac{1}{2}b\,D_{-2} - \frac{1}{2}a\,D_0, \\
\partial_\theta D_{+2} = +\frac{1}{2}b\,D_{+1} - \frac{1}{2}c\,D_{+3}, \quad \frac{-m-2\cos\theta}{\sin\theta}D_{+2} = -\frac{1}{2}b\,D_{+1} - \frac{1}{2}c\,D_{+3}, \\
\partial_\theta D_{-2} = +\frac{1}{2}c\,D_{-3} - \frac{1}{2}b\,D_{-1}, \quad \frac{-m+2\cos\theta}{\sin\theta}D_{-2} = -\frac{1}{2}c\,D_{-3} - \frac{1}{2}b;D_{-1}, \\
\sqrt{j(j+1)} = a, \quad \sqrt{(j-1)(j+2)} = b, \quad \sqrt{(j-2)(j+3)} = c.
\end{gathered} \tag{2}
$$

We use the following substitutions (which refer to the cyclic basis, the common multiplier $e^{-i\epsilon t}$ is omitted):

$$
H = h(r)D_0, \quad H_1 = \begin{vmatrix} h_0(r)\,D_0 \\ h_1(r)\,D_{-1} \\ h_2(r)\,D_0 \\ h_3(r)\,D_{+1} \end{vmatrix}, \quad H_2 = \begin{vmatrix} f_1(r)\,D_{-2} \\ f_2(r)\,D_0 \\ f_3(r)\,D_{+2} \\ c_1(r)\,D_{+1} \\ c_2(r)\,D_0 \\ c_3(r)\,D_{-1} \\ d_1(r)\,D_{-1} \\ d_2(r)\,D_0 \\ d_3(r)\,D_{+1} \\ f_0(r)\,D_0 \end{vmatrix},
$$

$$
\varphi_0 = \begin{vmatrix} E_{10}(r)D_{-1} \\ E_{20}(r)D_0 \\ E_{30}(r)D_{+1} \\ B_{10}(r)D_{+1} \\ B_{20}(r)D_0 \\ B_{30}(r)D_{-1} \end{vmatrix}, \varphi_1 = \begin{vmatrix} E_{11}(r)D_{-2} \\ E_{21}(r)D_{-1} \\ E_{31}(r)D_0 \\ B_{11}(r)D_0 \\ B_{21}(r)D_{-1} \\ B_{31}(r)D_{-2} \end{vmatrix}, \varphi_2 = \begin{vmatrix} E_{12}(r)D_{-1} \\ E_{22}(r)D_0 \\ E_{32}(r)D_{+1} \\ B_{12}(r)D_{+1} \\ B_{22}(r)D_0 \\ B_{32}(r)D_{-1} \end{vmatrix}, \varphi_3 = \begin{vmatrix} E_{13}(r)D_0 \\ E_{23}(r)D_{+1} \\ E_{33}(r)D_{+2} \\ B_{13}(r)D_{+2} \\ B_{23}(r)D_{+1} \\ B_{33}(r)D_0 \end{vmatrix}. \tag{3}
$$

In cyclic basis, the third projection of spin is a diagonal matrix. For special case $j = 0$, we should impose the following restrictions (because only the Wigner functions $D^0_{0,0} = 1$ may be used in describing such states):

$$
\begin{gathered}
h_1 = 0, h_3 = 0, \quad f_1 = 0, f_3 = 0, \quad c_1 = 0, c_3 = 0, \quad d_1 = 0, d_3 = 0, \\
E_{10} = 0, \quad E_{11} = 0, \quad E_{12} = 0, \quad E_{23} = 0, \\
E_{30} = 0, \quad E_{21} = 0, \quad E_{32} = 0, \quad E_{33} = 0, \\
B_{10} = 0, \quad B_{21} = 0, \quad B_{12} = 0, \quad B_{13} = 0, \\
B_{30} = 0, \quad B_{31} = 0, \quad B_{32} = 0, \quad B_{23} = 0.
\end{gathered} \tag{4}
$$

In order to take into account the presence of the Coulomb field, we make the change

$$
\epsilon \Rightarrow (\epsilon + \frac{\alpha}{r}) \equiv D_0,
$$

then the 39 equations (see in [1]) take on the form

H

$$\sqrt{2}a\left(h_1+h_3\right)+2r\left(iD_0h_0+mh+h_{2'}\right)+4h_2=0;$$

H_1

$$\sqrt{2}a\left(d_1+d_3\right)+2r\left(iD_0f_0+d_{2'}-3mh_0\right)-3iD_0rh+4d_2=0,$$

$$\frac{2ac_2-3ah+2bf_1}{6\sqrt{2}r}+\frac{1}{3}\left(iD_0d_1+c_{3'}+\frac{3c_3}{r}\right)-mh_1=0,$$

$$\sqrt{2}a\left(c_1+c_3\right)+r\left(2iD_0d_2+2f_{2'}+3h'-6mh_2\right)+4c_2+4f_2=0,$$

$$\frac{2ac_2-3ah+2bf_3}{6\sqrt{2}r}+\frac{1}{3}\left(iD_0d_3+c_{1'}+\frac{3c_1}{r}\right)-mh_3=0;$$

H_2

$$\frac{B_{31}}{r}-iD_0E_{11}-mf_1-\frac{b\left(B_{21}+2h_1\right)}{\sqrt{2}r}+B_{31}'=0,$$

$$4B_{11}-4B_{33}-8mrf_2+4\left(E_{20}-2h_2\right)+\sqrt{2}a\left(3B_{12}+B_{21}-B_{23}-3B_{32}+E_{10}+E_{30}-2\left(h_1+h_3\right)\right)$$
$$+2r\left(-iD_0\left(E_{13}+3E_{22}+E_{31}+2h_0\right)-B_{11}'+B_{33}'+E_{20}'+6h_{2'}\right)=0,$$

$$-2B_{13}+\sqrt{2}b\left(B_{23}-2h_3\right)-2r\left(iD_0E_{33}+mf_3+B_{13}'\right)=0,$$

$$\sqrt{2}bB_{13}+2B_{23}+\sqrt{2}a\left(B_{22}-B_{33}-2h_2\right)-4h_3-2r\left(2mc_1+iD_0\left(E_{23}+E_{32}\right)+B_{12}'-2h_{3'}\right)=0,$$

$$\sqrt{2}a\left(B_{12}+B_{21}-B_{23}-B_{32}-E_{10}-E_{30}-2\left(h_1+h_3\right)\right)$$
$$-2\left(2\left(E_{20}+2h_2\right)+r\left(4mc_2+iD_0\left(E_{13}+E_{22}+E_{31}-2h_0\right)+B_{11}'-B_{33}'+E_{20}'-2h_{2'}\right)\right)=0,$$

$$-4mrc_3+\sqrt{2}\left(aB_{11}-bB_{31}-a\left(B_{22}+2h_2\right)\right)-2\left(B_{21}+irD_0\left(E_{12}+E_{21}\right)+2h_1-r\left(B_{32}'+2h_{1'}\right)\right)=0,$$

$$-4mrd_1+\sqrt{2}\left(bE_{11}+a\left(-B_{20}+E_{31}-2h_0\right)\right)+2\left(B_{30}+E_{12}+2E_{21}+r\left(-iD_0\left(E_{10}+2h_1\right)+B_{30}'+E_{21}'\right)\right)=0,$$

$$\sqrt{2}a\left(B_{10}-B_{30}+E_{12}+E_{32}\right)+2\left(E_{13}+2E_{22}+E_{31}+r\left(-2md_2-iD_0\left(E_{20}+2h_2\right)+E_{22}'+2h_{0'}\right)\right)=0,$$

$$-2B_{10}+4E_{23}+2E_{32}+\sqrt{2}bE_{33}+\sqrt{2}a\left(B_{20}+E_{13}-2h_0\right)-2r\left(2md_3+iD_0\left(E_{30}+2h_3\right)+B_{10}'-E_{23}'\right)=0,$$

$$4B_{11}+\sqrt{2}a\left(B_{12}-B_{21}+B_{23}-B_{32}+3E_{10}+3E_{30}+2\left(h_1+h_3\right)\right)$$
$$+2\left(-2B_{33}+6E_{20}+4h_2+r\left(-4mf_0+iD_0\left(E_{13}-E_{22}+E_{31}-6h_0\right)+B_{11}'-B_{33}'+3E_{20}'+2h_{2'}\right)\right)=0;$$

H_3

$$\frac{6c_3+\sqrt{2}\left(ac_2+3af_0+bf_1\right)}{r}+2\left(-2iD_0d_1-3mE_{10}+c'_3\right)=0,$$

$$4c_2+\sqrt{2}a\left(c_1+c_3\right)+4f_2+2r\left(-2iD_0d_2-3\left(mE_{20}+f'_0\right)+f'_2\right)=0,$$

$$\frac{6c_1+\sqrt{2}\left(ac_2+3af_0+bf_3\right)}{r}+2\left(-2iD_0d_3-3mE_{30}+c'_1\right)=0,$$

$$-mB_{10}+\frac{ad_2}{\sqrt{2}r}+\frac{d_3}{r}+d_{3'}=0,\quad \frac{a\left(d_3-d_1\right)}{\sqrt{2}r}-mB_{20}=0,$$

$$d_1+\sqrt{2}ad_2+2r\left(mB_{30}+d'_1\right)=0,\quad \frac{bd_1}{\sqrt{2}r}-mE_{11}-iD_0f_1=0,\quad -iD_0c_3-mE_{21}-d'_1=0,$$

$$2d_2+\sqrt{2}a\left(2d_1-d_3\right)-2r\left(3mE_{31}+iD_0\left(3c_2+f_0\right)+d'_2\right)=0,$$

$$\sqrt{2}a\left(c_1-2c_3\right)-2\left(c_2+f_2\right)+2r\left(3mB_{11}+iD_0d_2-3c'_2+f'_2\right)=0,$$

$$-3mB_{21}+iD_0d_1+\frac{\sqrt{2}\left(2ac_2-bf_1\right)}{r}+c'_3=0,$$

$$\sqrt{2}bc_3+2f_1+2r\left(mB_{31}+f'_1\right)=0,\quad -iD_0c_3+\frac{d_1}{r}+\frac{ad_2}{\sqrt{2}r}-mE_{12}=0,$$

$$4d_2+\sqrt{2}a\left(d_1+d_3\right)-6mrE_{22}+2irD_0\left(f_0-3f_2\right)-4rd'_2=0,$$

$$-iD_0c_1+\frac{ad_2}{\sqrt{2}r}+\frac{d_3}{r}-mE_{32}=0,$$

$$6mrB_{12}-6c_1+2irD_0d_3+\sqrt{2}\left(ac_2-3af_2+bf_3\right)-4rc'_1=0,$$

$$\frac{a\left(c_1-c_3\right)}{\sqrt{2}r}-mB_{22}=0,$$

$$-6mrB_{32}-6c_3+2irD_0d_1+\sqrt{2}\left(ac_2+bf_1-3af_2\right)-4rc'_3=0,$$

$$-2d_2+\sqrt{2}a\left(d_1-2d_3\right)+2r\left(3mE_{13}+iD_0\left(3c_2+f_0\right)+d'_2\right)=0,$$

$$-iD_0c_1-mE_{23}-d_{3'}=0,\quad \frac{bd_3}{\sqrt{2}r}-mE_{33}-iD_0f_3=0,$$

$$-mB_{13}+\frac{bc_1}{\sqrt{2}r}+\frac{f_3}{r}+f'_3=0,\quad -3mB_{23}-iD_0d_3+\frac{\sqrt{2}\left(bf_3-2ac_2\right)}{r}-c'_1=0,$$

$$\sqrt{2}a\left(c_3-2c_1\right)-2\left(c_2+f_2\right)+2r\left(-3mB_{33}+iD_0d_2-3c'_2+f'_2\right)=0.$$

Taking in mind restrictions (4), we obtain more simple system

H

$$2irD_0h_0+2mrh+2rh_{2'}+4h_2=0\,;$$

H_1

$$2irD_0f_0-3irD_0h+2rd_{2'}+4d_2-6mrh_0=0,$$

$$\frac{ac_2}{3\sqrt{2}r}-\frac{ah}{2\sqrt{2}r}=0,\qquad 0=0\quad(a\equiv 0)$$

$$2irD_0d_2+4c_2+2rf'_2+4f_2+3rh'-6mrh_2=0,$$

$$\frac{ac_2}{3\sqrt{2}r}-\frac{ah}{2\sqrt{2}r}=0,\qquad 0=0\quad(a\equiv 0)$$

H_2

$$-\frac{bB_{21}}{\sqrt{2}r}=0,\qquad 0=0\quad(B_{21}\equiv 0)$$

$$-2irD_0E_{13}-6irD_0E_{22}-2irD_0E_{31}-4irD_0h_0+2rB_{33}'-4B_{33}+2rE_{20}'+4E_{20}-8mrf_2+12rh'_2-8h_2=0,$$

$$0=0,$$

$$\sqrt{2}aB_{22}-\sqrt{2}aB_{33}-2\sqrt{2}ah_2=0,\quad 0=0\quad(a\equiv 0),$$

$$-2irD_0E_{13}-2irD_0E_{22}-2irD_0E_{31}+4irD_0h_0+2rB_{33}'-8mrc_2-2rE_{20}'-4E_{20}+4rh'_2-8h_2=0,$$

$$-\sqrt{2}aB_{22}-2\sqrt{2}ah_2-2B_{21}=0,\quad 0=0\quad(a\equiv 0,B_{21}\equiv 0)),$$

$$-\sqrt{2}aB_{20}+\sqrt{2}aE_{31}-2\sqrt{2}ah_0=0,\quad 0=0\quad(a\equiv 0),$$

$$-2irD_0E_{20}-4irD_0h_2-4mrd_2+2rE_{22}'+2E_{13}+4E_{22}+2E_{31}+4rh_{0'}=0,$$

$$\sqrt{2}aB_{20}+\sqrt{2}aE_{13}-2\sqrt{2}ah_0=0,\quad 0=0\quad (a\equiv 0),$$

$$+2irD_0E_{13}-2irD_0E_{22}+2irD_0E_{31}-12irD_0h_0-2rB_{33}'-4B_{33}+6rE_{20}'+12E_{20}-8mrf_0+4rh'_2+8h_2=0,$$

H_3

$$\frac{\sqrt{2}ac_2}{r}+\frac{3\sqrt{2}af_0}{r}=0,\quad 0=0\quad (a\equiv 0),$$

$$-4irD_0d_2+4c_2-6mrE_{20}-6rf'_0+2rf'_2+4f_2=0,$$

$$\frac{\sqrt{2}ac_2}{r}+\frac{3\sqrt{2}af_0}{r}=0,\quad 0=0\quad (a\equiv 0),$$

$$\frac{ad_2}{\sqrt{2}r}=0,\quad 0=0\quad (a\equiv 0),\quad -mB_{20}=0,\quad 0=0\quad (B_{20}\equiv 0)$$

$$\sqrt{2}ad_2=0,\quad 0=0\quad (a\equiv 0),$$

$$-6irD_0c_2-2irD_0f_0-2rd'_2+2d_2-6mrE_{31}=0,$$

$$2irD_0d_2-6rc'_2-2c_2+2rf'_2-2f_2+6mrB_{11}=0,$$

$$\frac{2\sqrt{2}ac_2}{r}=0,\quad 0=0\quad (a\equiv 0),$$

$$\frac{ad_2}{\sqrt{2}r}=0,\quad 0=0\quad (a\equiv 0),\quad 2iD_0rf_0-6irD_0f_2-4rd_{2'}+4d_2-6mrE_{22}=0,$$

$$\frac{ad_2}{\sqrt{2}r}=0,\quad 0=0\quad (a\equiv 0),\quad \sqrt{2}ac_2-3\sqrt{2}af_2=0,\quad 0=0\quad (a\equiv 0),$$

$$-mB_{22}=0,\qquad B_{22}=0,\quad \sqrt{2}ac_2-3\sqrt{2}af_2=0,\quad 0=0\quad (a\equiv 0),$$

$$6irD_0c_2+2irD_0f_0+2rd'_2-2d_2+6mrE_{13}=0,$$

$$-\frac{2\sqrt{2}ac_2}{r}=0,\quad 0=0\quad (a\equiv 0),$$

$$2irD_0d_2-6mrB_{33}-6rc_{2'}-2c_2+2rf'_2-2f_2=0.$$

Whence it follows

H

$$2irD_0h_0+2mrh+2rh'_2+4h_2=0,$$

H_1

$$2irD_0f_0-3irD_0h+2rd'_2+4d_2-6mrh_0=0,$$

$$2irD_0d_2+4c_2+2rf'_2+4f_2+3rh'-6mrh_2=0,$$

H_2

$$-2irD_0E_{13}-6irD_0E_{22}-2irD_0E_{31}-4irD_0h_0+2rB_{33}'-4B_{33}+2rE_{20}'+4E_{20}-8mrf_2+12rh'_2-8h_2=0,$$

$$-2irD_0E_{13}-2irD_0E_{22}-2irD_0E_{31}+4irD_0h_0+2rB_{33}'-8mrc_2-2rE_{20}'-4E_{20}+4rh'_2-8h_2=0,$$

$$-2irD_0E_{20}-4irD_0h_2-4mrd_2+2rE_{22}'+2E_{13}+4E_{22}+2E_{31}+4rh'_0=0,$$

$$+2irD_0E_{13}-2irD_0E_{22}+2irD_0E_{31}-12irD_0h_0-2rB_{33}'-4B_{33}+6rE_{20}'+12E_{20}-8mrf_0+4rh'_2+8h_2=0,$$

H_3

$$-4irD_0d_2+4c_2-6mrE_{20}-6rf'_0+2rf'_2+4f_2=0,$$

$$-6irD_0c_2 - 2irD_0f_0 - 2rd'_2 + 2d_2 - 6mrE_{31} = 0,$$
$$2irD_0d_2 - 6rc'_2 - 2c_2 + 2rf'_2 - 2f_2 + 6mrB_{11} = 0,$$
$$2irD_0f_0 - 6irD_0f_2 - 4rd'_2 + 4d_2 - 6mrE_{22} = 0,$$
$$6irD_0c_2 + 2irD_0f_0 + 2rd'_2 - 2d_2 + 6mrE_{13} = 0,$$
$$2irD_0d_2 - 6mrB_{33} - 6rc_{2'} - 2c_2 + 2rf'_2 - 2f_2 = 0.$$

Let us take into account the parity restrictions [1]: $P = (-1)^{j+1}$

$$h = 0, \quad h_0 = 0, f_0 = 0, \quad h_2 = 0, f_2 = 0, c_2 = 0, d_2 = 0,$$
$$E_{20} = 0, E_{22} = 0, E_{13} = -E_{31}, B_{33} = B_{11}; \tag{5}$$

the the above system simplifies

$$H \quad 0 = 0,$$
$$H_1 \qquad 0 = 0,$$
$$H_2 \qquad 2rB_{11}' - 4B_{11} = 0, \quad 2rB_{11}' = 0, \quad 0 = 0, \quad -2rB_{11}' - 4B_{11} = 0,$$
$$H_3 \qquad 0 = 0, \quad E_{31} = 0, \quad 6mrB_{11} = 0, \quad 0 = 0, \quad 6mrE_{13} = 0, \quad -6mrB_{33} = 0.$$

We can readily see that any states with this parity do not exist.

For the parity $P = (-1)^j$, we have the following restrictions [1]:

$$h_3 = +h_1; \quad f_3 = +f_1, \quad c_3 = +c_1, \quad d_3 = +d_1;$$

and the system of equations reads

H

$$2irD_0h_0 + 2mrh + 2rh'_2 + 4h_2 = 0$$

H_1

$$2irD_0f_0 - 3irR_0h + 2rd'_2 + 4d_2 - 6mrh_0 = 0,$$
$$2irR_0d_2 + 4c_2 + 2rf'_2 + 4f_2 + 3rh' - 6mrh_2 = 0,$$

H_2

$$-6irD_0E_{22} - 4irD_0E_{31} - 4irD_0h_0 - 4rB_{11}' + 8B_{11} + 2rE_{20}' + 4E_{20} - 8mrf_2 + 12rh'_2 - 8h_2 = 0,$$
$$-2irD_0E_{22} - 4irD_0E_{31} + 4irD_0h_0 - 4rB_{11}' - 8mrc_2 - 2rE_{20}' - 4E_{20} + 4rh'_2 - 8h_2 = 0,$$
$$-2irD_0E_{20} - 4irD_0h_2 - 4mrd_2 + 2rE_{22}' + 4E_{22} + 4E_{31} + 4rh'_0 = 0,$$
$$-2irD_0E_{22} + 4irD_0E_{31} - 12irD_0h_0 + 4rB_{11}' + 8B_{11} + 6rE_{20}' + 12E_{20} - 8mrf_0 + 4rh'_2 + 8h_2 = 0,$$

H_3

$$-4irD_0d_2 + 4c_2 - 6mrE_{20} - 6rf_{0'} + 2rf'_2 + 4f_2 = 0,$$
$$-6irD_0c_2 - 2irD_0f_0 - 2rd'_2 + 2d_2 - 6mrE_{31} = 0,$$
$$2irD_0d_2 + 6mrB_{11} - 6rc'_2 - 2c_2 + 2rf'_2 - 2f_2 = 0,$$
$$2irD_0f_0 - 6irR_0f_2 - 4rd'_2 + 4d_2 - 6mrE_{22} = 0,$$
$$6irD_0c_2 + 2irD_0f_0 + 2rd'_2 - 2d_2 + 6mrE_{31} = 0, \textit{ the same}$$
$$2irD_0d_2 + 6mrB_{11} - 6rc'_2 - 2c_2 + 2rf'_2 - 2f_2 = 0, \textit{ the same}$$

We express the variables h_0, h_2 from the group H_1, and the variables $E_{20}, E_{31}, B_{11}, E_{22}$ from the group H_3, and substitute the results in the equations of the groups H and H_2:

$$\frac{2}{3}i\frac{1}{m^2}D_0D_rd_2+\frac{2}{3}i\frac{1}{m^2}D_rD_0d_2+\frac{8}{3}i\frac{1}{m^2r}D_0d_2+\frac{4}{3}\frac{1}{m^2r}D_rc_2+\frac{4}{3}\frac{1}{m^2r^2}c_2+$$
$$+\frac{8}{3}\frac{1}{m^2r}D_rf_2+\frac{2}{3}\frac{1}{m^2}D_r^2f_2+\frac{4}{3}\frac{1}{m^2r^2}f_2+D_0^2\frac{1}{m^2}h+D_r^2\frac{1}{m^2}h+2D_r\frac{1}{m^2r}h+2h-\frac{2}{3}\frac{1}{m^2}D_0^2f_0=0,$$

$$+4i\frac{1}{m^2}D_rD_0d_2+4i\frac{1}{m^2}D_0D_rd_2-16i\frac{1}{m^2r}D_0d_2+2\frac{1}{m^2}D_0^2f_0-4\frac{1}{m^2r}D_rf_0-2\frac{1}{m^2}D_r^2f_0-$$
$$-6\frac{1}{m^2}D_0^2f_2-8f_2+4\frac{1}{m^2r}D_rf_2+6\frac{1}{m^2}D_r^2f_2-8\frac{1}{m^2r^2}f_2-2\frac{1}{m^2}D_0^2h+6\frac{1}{m^2}D_r^2h-4\frac{1}{m^2r}D_rh-$$
$$-4\frac{1}{m^2}D_0^2c_2+16\frac{1}{m^2r}D_rc_2-4\frac{1}{m^2}D_r^2c_2-8\frac{1}{m^2r^2}c_2=0,$$

$$-4\frac{1}{m^2}D_0^2c_2-8c_2-4\frac{1}{m^2}D_r^2c_2-\frac{1}{m^2r^2}8c_2+4i\frac{1}{m^2}D_0D_rd_2+4i\frac{1}{m^2}D_rD_0d_2-$$
$$-2\frac{1}{m^2}D_0^2f_0+4\frac{1}{m^2r}D_rf_0+2\frac{1}{m^2}D_r^2f_0-2\frac{1}{m^2}D_0^2f_2-4\frac{1}{m^2r}D_rf_2+2\frac{1}{m^2}D_r^2f_2-8\frac{1}{m^2r^2}f_2+$$
$$+2D_0^2\frac{1}{m^2}h+2D_r^2\frac{1}{m^2}h-4\frac{1}{m^2r}D_rh=0,$$

$$-4d_2-8i\frac{1}{m^2r}D_0c_2+2i\frac{1}{m^2}D_0D_rf_0+2i\frac{1}{m^2}D_rD_0f_0-$$
$$-2i\frac{1}{m^2}D_0D_rf_2-8i\frac{1}{m^2r}D_0f_2-2i\frac{1}{m^2}D_rD_0f_2-2i\frac{1}{m^2}D_0D_rh-2i\frac{1}{m^2}D_rD_0h=0,$$

$$4\frac{1}{m^2}D_0^2c_2+16\frac{1}{m^2r}D_rc_2+4\frac{1}{m^2}D_r^2c_2+8\frac{1}{m^2r^2}c_2-4i\frac{1}{m^2}D_0D_rd_2-16i\frac{1}{m^2r}D_0d_2-4i\frac{1}{m^2}D_rD_0d_2+$$
$$+6\frac{1}{m^2}D_0^2f_0-8f_0-12\frac{1}{m^2r}D_rf_0-6\frac{1}{m^2}D_r^2f_0-2\frac{1}{m^2}D_0^2f_2+12\frac{1}{m^2r}D_rf_2+2\frac{1}{m^2}D_r^2f_2+8\frac{1}{m^2r^2}f_2-$$
$$-6\frac{1}{m^2}D_0^2h+2\frac{1}{m^2}D_r^2h+4\frac{1}{m^2r}D_rh=0.$$

For states with the parity $P=(-1)^j$, the 5 independent components are divided into large (L) and small (S) parts as follows (see in [1])

$$h=S_1,\quad f_2=L_2+S_2,\quad c_2=\frac{L_2+S_2}{2},\quad d_2=S_{10},\quad f_0=S_{12}. \tag{6}$$

When performing the non-relativistic approximation, we should separate the rest energy by formal change $D_0\Rightarrow M+iD_0, M>0$; besides, bellow we should take into account the identify $m=-M$.

As known, we should assume the following smallness orders for the wave function components and derivatives (see in [1])

$$L\sim 1,\quad S_k\sim x,\quad \frac{1}{m}D_r\sim x,\quad \frac{1}{mr}\sim x,$$
$$\frac{1}{m}iD_0\Rightarrow\frac{1}{m}(m+iD_0)=\frac{1}{m}[m+(E+\frac{\alpha}{r})]\sim(1+\frac{1}{m}R_0)\sim(1+x^2),$$
$$\frac{1}{m^2}D_0^2=(-1-\frac{2}{m}(E+\frac{\alpha}{r})-\frac{1}{m^2}(E+\frac{\alpha}{r})^2)=(-1-2\frac{1}{m}R_0)-\frac{1}{m^2}R_0^2\sim((-1-x^2)-x^4).$$
$$\frac{2}{3}\frac{1}{m}(1+\frac{1}{m}R_0)D_rS_{10}+\frac{2}{3}\frac{1}{m}D_r(1+\frac{1}{m}R_0)S_{10}+\frac{8}{3}\frac{1}{mr}(1+\frac{1}{m}R_0)S_{10}+$$

$$+\frac{4}{3}\frac{1}{m^2 r}D_r\frac{L_2+S_2}{2}+\frac{4}{3}\frac{1}{m^2 r^2}\frac{L_2+S_2}{2}+$$
$$+\frac{8}{3}\frac{1}{m^2 r}D_r(L_2+S_2)+\frac{2}{3}\frac{1}{m^2}D_r^2(L_2+S_2)+\frac{4}{3}\frac{1}{m^2 r^2}(L_2+S_2)+$$
$$+(-1-2\frac{1}{m}R_0)S_1+D_r^2\frac{1}{m^2}S_1+2D_r\frac{1}{m^2 r}S_1+2S_1-\frac{2}{3}\frac{1}{m^2}D_0^2 S_{12}=0,$$

$$+4\frac{1}{m}D_r(1+\frac{1}{m}R_0)S_{10}+4\frac{1}{m}(1+\frac{1}{m}R_0)D_r S_{10}-16\frac{1}{mr}(1+\frac{1}{m}R_0)S_{10}+$$
$$+2(-1-2\frac{1}{m}R_0)S_{12}-4\frac{1}{m^2 r}D_r S_{12}-2\frac{1}{m^2}D_r^2 S_{12}-$$
$$-6(-1-2\frac{1}{m}R_0)(L_2+S_2)-8(L_2+S_2)+4\frac{1}{m^2 r}D_r(L_2+S_2)+6\frac{1}{m^2}D_r^2(L_2+S_2)-$$
$$-8\frac{1}{m^2 r^2}(L_2+S_2)-2(-1-2\frac{1}{m}R_0)S_1+6\frac{1}{m^2}D_r^2 S_1-4\frac{1}{m^2 r}D_r S_1-$$
$$-4(-1-2\frac{1}{m}R_0)\frac{L_2+S_2}{2}+16\frac{1}{m^2 r}D_r\frac{L_2+S_2}{2}-4\frac{1}{m^2}D_r^2\frac{L_2+S_2}{2}-8\frac{1}{m^2 r^2}\frac{L_2+S_2}{2}=0,$$

$$-4(-1-2\frac{1}{m}R_0)\frac{L_2+S_2}{2}-8\frac{L_2+S_2}{2}-4\frac{1}{m^2}D_r^2\frac{L_2+S_2}{2}-\frac{1}{m^2 r^2}8\frac{L_2+S_2}{2}+$$
$$+4\frac{1}{m}(1+\frac{1}{m}R_0)D_r S_{10}+4\frac{1}{m}D_r(1+\frac{1}{m}R_0)S_{10}-$$
$$-2(-1-2\frac{1}{m}R_0)S_{12}+4\frac{1}{m^2 r}D_r S_{12}+2\frac{1}{m^2}D_r^2 S_{12}-2(-1-2\frac{1}{m}R_0)(L_2+S_2)-$$
$$-4\frac{1}{m^2 r}D_r(L_2+S_2)+2\frac{1}{m^2}D_r^2(L_2+S_2)-8\frac{1}{m^2 r^2}(L_2+S_2)+$$
$$+2D_0^2\frac{1}{m^2}S_1+2D_r^2\frac{1}{m^2}S_1-4\frac{1}{m^2 r}D_r S_1=0,$$

$$-4S_{10}-8\frac{1}{mr}(1+\frac{1}{m}R_0)\frac{L_2+S_2}{2}+2\frac{1}{m}(1+\frac{1}{m}R_0)D_r S_{12}+2\frac{1}{m}D_r(1+\frac{1}{m}R_0)S_{12}-$$
$$-2\frac{1}{m}(1+\frac{1}{m}R_0)D_r(L_2+S_2)-8\frac{1}{mr}(1+\frac{1}{m}R_0)(L_2+S_2)-2\frac{1}{m}D_r(1+\frac{1}{m}R_0)(L_2+S_2)-$$
$$-2\frac{1}{m}(1+\frac{1}{m}R_0)D_r S_1-2\frac{1}{m}D_r(1+\frac{1}{m}R_0)S_1=0,$$

$$4(-1-2\frac{1}{m}R_0)\frac{L_2+S_2}{2}+16\frac{1}{m^2 r}D_r\frac{L_2+S_2}{2}+4\frac{1}{m^2}D_r^2\frac{L_2+S_2}{2}+8\frac{1}{m^2 r^2}\frac{L_2+S_2}{2}-$$
$$-4\frac{1}{m}(1+\frac{1}{m}R_0)D_r S_{10}-16\frac{1}{mr}(1+\frac{1}{m}R_0)S_{10}-4\frac{1}{m}D_r(1+\frac{1}{m}R_0)S_{10}+$$
$$+6(-1-2\frac{1}{m}R_0)S_{12}-8S_{12}-12\frac{1}{m^2 r}D_r S_{12}-6\frac{1}{m^2}D_r^2 S_{12}-2\frac{1}{m^2}D_0^2(L_2+S_2)+$$
$$+12\frac{1}{m^2 r}D_r(L_2+S_2)+2\frac{1}{m^2}D_r^2(L_2+S_2)+8\frac{1}{m^2 r^2}(L_2+S_2)-$$
$$-6(-1-2\frac{1}{m}R_0)S_1+2\frac{1}{m^2}D_r^2 S_1+4\frac{1}{m^2 r}D_r S_1=0.$$

We will preserve only the terms of the primary orders x^0, x, x^2, equating them to zero; in this way we obtain

x^1

$$S_1+\frac{2S_{12}}{3}=0,\quad S_1-S_{12}=0,\quad 2S_{12}-2S_1=0,$$
$$-\frac{D_rL_2}{m}-S_{10}-\frac{3L_2}{mr}=0,\quad 6S_1-14S_{12}=0,$$

x^2

$$\frac{D_r^2L_2}{3m^2}+\frac{5D_rL_2}{3m^2r}+\frac{2D_rS_{10}}{3m}+\frac{L_2}{m^2r^2}+\frac{4S_{10}}{3mr}=0,$$
$$\frac{D_r^2L_2}{m^2}+\frac{3D_rL_2}{m^2r}+\frac{2D_rS_{10}}{m}-\frac{3L_2}{m^2r^2}+\frac{4R_0L_2}{m}-\frac{4S_{10}}{mr}=0,$$
$$-\frac{D_rL_2}{m^2r}+\frac{2D_rS_{10}}{m}-\frac{3L_2}{m^2r^2}+\frac{2R_0L_2}{m}=0,$$
$$-\frac{D_rS_1}{m}-\frac{D_rS_2}{m}+\frac{D_rS_{12}}{m}-\frac{3S_2}{mr}=0,\quad \textit{not needed}$$
$$\frac{D_r^2L_2}{m^2}+\frac{5D_rL_2}{m^2r}-\frac{2D_rS_{10}}{m}+\frac{3L_2}{m^2r^2}-\frac{4S_{10}}{mr}=0,$$

From equations related to order x, it follows

$$S_1=S_{12}=0,\quad S_{10}=-\frac{D_rL_2}{m}-\frac{3L_2}{mr};\tag{7}$$

substituting these into equations related to order x^2, we get

$$-\frac{D_r^2L_2}{3m^2}-\frac{5D_rL_2}{3m^2r}-\frac{L_2}{m^2r^2}=0,\quad -\frac{D_r^2L_2}{m^2}+\frac{D_rL_2}{m^2r}+\frac{15L_2}{m^2r^2}+\frac{4R_0L_2}{m}=0,$$
$$-\frac{2D_r^2L_2}{m^2}-\frac{7D_rL_2}{m^2r}+\frac{3L_2}{m^2r^2}+\frac{2R_0L_2}{m}=0,\quad \frac{D_r^2L_2}{m^2}+\frac{5D_rL_2}{m^2r}+\frac{3L_2}{m^2r^2}=0.$$

The first and the last equations coincide, so we have only three different equations (recall that $R_0=D_0$):

$$+\frac{4D_0L_2}{m}-\frac{D_r^2L_2}{m^2}+\frac{D_rL_2}{m^2r}+\frac{15L_2}{m^2r^2}=0,\quad +\frac{2D_0L_2}{m}-\frac{2D_r^2L_2}{m^2}-\frac{7D_rL_2}{m^2r}+\frac{3L_2}{m^2r^2}=0,$$
$$\frac{D_r^2L_2}{m^2}+\frac{5D_rL_2}{m^2r}+\frac{3L_2}{m^2r^2}=0.$$

Expressing the term $3L_2/m^2r^2$ from the last equation, and substituting this result in two remaining equations, we obtain one the same equation

$$\frac{2D_0}{m}L_2-\frac{3D_r^2}{m^2}L_2-\frac{12D_r}{m^2r}L_2=0\,.$$

Let us take into account identities $D_0=E+\alpha/r, m=-M$; then we arrive at

$$[\frac{d^2}{dr^2}+\frac{4}{r}\frac{d}{dr}+\frac{2ME}{3}+\frac{2M\alpha}{3}\frac{1}{r}]L_2=0\,.\tag{8}$$

Near the point $r=0$, solutions behave as (let $L_2(r)=f(r)$) $f=r^A, A=0, A=-3$; to bound states may correspond only the value $A=0$. At infinity, we get

$$x=\frac{1}{r},\quad (\frac{d^2}{dx^2}-\frac{6}{x}\frac{d}{dx}+\frac{2ME}{3}\frac{1}{x^4}+\frac{2M\alpha}{3}\frac{1}{x})f=0\,.$$

Thus, we have an equation of the confluent hypergeometric type (the point $r=0$ is regular, and the point $r=-\infty$ is irregular of the rank 2). Let us introduce the shortening notations (remembering that $m=-M<0$)

$$\frac{2mE}{3}=a,\quad \frac{2m\alpha}{3}=b,\quad L_2=f,\quad [\frac{d^2}{dr^2}+\frac{4}{r}\frac{d}{dr}-a-\frac{b}{r}]f=0\,.$$

Using the substitution $f=r^A e^{Br}F$, we obtain

$$F''+(\frac{2A}{r}+\frac{4}{r}+2B)F'+[\frac{A(A-1)+4A}{r^2}+(B^2-a)+\frac{2AB+4B-b}{r}]F=0\,.$$

Let us impose restrictions (relevant to bound states)

$$A(A-1)+4A=0\ \Rightarrow\ A=0;\quad (B^2-a)=0\ \Rightarrow\ B=-\sqrt{a}=-\sqrt{\frac{2mE}{3}}\quad (E<0); \tag{9}$$

then the equation simplifies

$$rF''+(2A+4+2Br)F'+(2AB+4B-b)F=0.$$

In the variable $x=-2Br$, it reads

$$x\frac{d^2}{dx^2}f+(2A+4-x)\frac{d}{dx}f-(A+2-\frac{b}{2B})F=0\,.$$

The polynomial condition leads to the following quantization rule

$$a=A+2-\frac{b}{2B}=-n,\quad 2+n=\frac{b}{2B},\quad where\quad B=-\sqrt{\frac{-2ME}{3}},\quad b=\frac{-2M\alpha}{3};$$

so the energy spectrum for states with $j=0$ is given by the formula

$$E_n=-\frac{M\alpha^2}{6}\frac{1}{(2+n)^2},n=0,1,2,.... \tag{10}$$

Conclusion

In the previous study of the problem of a non-relativistic spin 2 particle in external Coulomb field, the case of minimal value of the quantum number $j=0$ was not considered. In the present paper, this special class of states has been invesigated, and the corresponding wave function and the energy spectrum are found. Recall that for states with the quantum numbers $j=1,2,...$ exist five types of solutions and correspondingly five energy spectra [1].

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Рукапіс паступіў у рэдакцыю 16.03.2026