ФІЗІКА

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PAULI EQUATION IN CURVILINEAR COORDINATES OF MINKOWSKI SPACE, SEPARATING THE VARIBLES

The non-relativistic Pauli approximation for a spin ½ particle is studied in arbitatry curvilinear orthogonal coordinates.The final goal is to develop ф unified approach to separating the variables in equation for s spin ½ particle in all 12 system of orthogonal coordinates in the flat space, first in more simple Pauli approximation. A general structure for the covariant equation is derived Pauli equation in arbitrary orthogonal coordinates, it includes the relevant tetrads and Ricci rotation coefficients. Because the possibility to develop a unified approach to separating the variables in ordinary presentation of the covariant Pauli equation is rather problematitic, other way for studying the problem is proposed. It is based on transforming the usual 2-component Pauli equation to an equivalent system of first order four differential equations, by introducing two auxiliary components, because it is known that the task of separating the variables in the system of differential equations in partial derivatives may be solved easier for the first order systems. One simple example for illustating this approach, is given; a spin ½ particle problem in presence of external magnetic field when using the cylindrical coordinates.

Key words: *spin ½ particle, curvilinear coordinates, tetrad formalism, Ricci rotation coefficients, Pauli approximation, separation of the variables.*

УРАВНЕНИЕ ПАУЛИ В КРИВОЛИНЕЙНЫХ КООРДИНАТАХ ПРОСТРАНСТВА МИНКОВСКОГО, РАЗДЕЛЕНИЕ ПЕРЕМЕННЫХ

Нерелятивистское приближение Паули для частицы со спином ½ изучается в произвольных криволинейных ортогональных координатах. Конечной целью является разработка единого подхода к разделению переменных в уравнении для частицы со спином ½ во всех 12 системах ортогональных координат в плоском пространстве, сначала в более простом приближении Паули. Выведена общая структура для ковариантного уравнения Паули в произвольных ортогональных координатах, которая включает в себя соответствующие тетрады и коэффициенты вращения Риччи. Поскольку возможность разработки единого подхода к разделению переменных в обычном представлении ковариантного уравнения Паули довольно проблематична, предложен другой способ изучения проблемы. Он основан на преобразовании обычного 2-компонентного уравнения Паули в эквивалентную систему из четырех дифференциальных уравнений первого порядка путем введения двух вспомогательных компонентов, поскольку известно, что задача разделения переменных в системе дифференциальных уравнений в частных производных может быть решена проще для систем первого порядка. Приведен простой пример, иллюстрирующий этот подход: задача о частице со спином ½ в присутствии внешнего магнитного поля при использовании цилиндрических координат.

Ключевые слова: *частица со спином ½, криволинейные координаты, тетрадный формализм, коэффициеты вращения Риччи, приближение Паули, разделение переменных.*

Introduction

The study of all possibilities for separating the variables in quantum mechanical equations is important for applications in theoretical problems.

In particular, for flat Euclidean space there is known 13 system of coordinates allowing for the complete separation of the variables in the Laplace equation, closely related to Schrödinger equation.

This problem becomes more involved when turning to the Dirac equation or Maxwell equation, and other.

However, in [1] it was shown that with the use of squaring procedure for the Dirac equation leading to the scalar Klein – Fock – Gordon equation, one may construct solutions for the Dirac equation as well, using the known solutions for the scalar equation.

Therefore, the possibility to get solutions for Klein – Fock – Gordon equation equation in all 13 coordinates systems, permits us to expectcorresponding solutions for the Dirac equation as well. Similar posibility exists also for Maxwell theory [2; 3].

In the present paper, we will derive an explicit form of Pauli equation in arbitrary orthogonal coordinates of the 3-dimensional Euclidean space, parameterized by the metric

$$dS^2 = c^2 dt^2 - h_1^2(x)(dx^1)^2 - h_2^2(x)(dx^2)^2 - h_3^2(x)(dx^3)^2 . \tag{1}$$

We will start with the covariant form of the Dirac equation [4]

$$[i\gamma^c (e^{\alpha}_{(c)} \frac{\partial}{\partial x^{\alpha}} + \frac{1}{2}\gamma_{[ab]c} j^{[ab]} - \frac{mc}{\hbar}]\Psi = 0, \tag{2}$$

where $j^{[ab]}$ stand for six Loretzian generators for bispinor representation of the Lorentz group; $e^{\alpha}_{(c)}(x)$ stands for the digonal tetrad, related to the metric (1).

Pauli approximation in orthogonal coordinates for E_3-space

Let us start with the relativistic Dirac equation in orthogonal coordinates

$$\{i\gamma^0(\partial_0 + A_0) + i\gamma^1(\frac{1}{h_1}(\partial_1 + ieA_1) + G_{11} j^{23} + G_{21} j^{31} + G_{31} j^{12}) +$$

$$+ i\gamma^2(\frac{1}{h_2}(\partial_2 + ieA_2) + G_{12} j^{23} + G_{22} j^{31} + G_{32} j^{12}) +$$

$$+ i\gamma^3(\frac{1}{h_3}(\partial_3 + ieA_3) + G_{13} j^{23} + G_{23} j^{31} + G_{33} j^{12}) - M\}\Psi = 0; \tag{3}$$

where the shortening notations for the Ricci coefficients are used

$$G_{11} = \gamma_{011} = \frac{\partial_0 h_1}{h_1}, \quad G_{12} = \gamma_{232} = -\frac{\partial_3 h_2}{h_2 h_3}, \quad G_{13} = \gamma_{233} = +\frac{\partial_2 h_3}{h_2 h_3},$$

$$G_{21} = \gamma_{311} = +\frac{\partial_3 h_1}{h_1 h_3}, \quad G_{22} = \gamma_{022} = \frac{\partial_0 h_2}{h_2}, \quad G_{23} = \gamma_{313} = -\frac{\partial_1 h_3}{h_1 h_3},$$

$$G_{31} = \gamma_{121} = -\frac{\partial_2 h_1}{h_1 h_2}, \quad G_{32} = \gamma_{122} = +\frac{\partial_1 h_2}{h_1 h_2}, \quad G_{33} = \gamma_{033} = \frac{\partial_0 h_3}{h_3};$$

for static metrics, the formulas simplify due to identities $G_{11} = G_{22} = G_{33} = 0$.

It is convenient to apply the shortening notations for derivatives

$$D_1 = \frac{1}{h_1}(\partial_1 + ieA_1) + G_{11} j^{23} + G_{21} j^{31} + G_{31} j^{12},$$

$$D_2 = \frac{1}{h_2}(\partial_2 + ieA_2) + G_{12} j^{23} + G_{22} j^{31} + G_{32} j^{12},$$

$$D_3 = \frac{1}{h_3}(\partial_1 + ieA_3) + G_{13} j^{23} + G_{23} j^{31} + G_{33} j^{12}.$$

We will use the Pauli presentation for Dirac matrices:

$$\gamma^0 = \begin{vmatrix} I & 0 \\ 0 & -I \end{vmatrix}, \gamma^1 = \begin{vmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{vmatrix}, \gamma^2 = \begin{vmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{vmatrix}, \gamma^3 = \begin{vmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{vmatrix},$$

$$\sigma_0 = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}, \quad \sigma_1 = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}, \quad \sigma_2 = \begin{vmatrix} 0 & -i \\ i & 0 \end{vmatrix}, \quad \sigma_3 = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix};$$

also we need the bispinor generators $j^{kl} = \frac{1}{4}(\gamma^k \gamma^l - \gamma^l \gamma^k)$:

$$j^{23} = -\frac{i}{2}\begin{vmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{vmatrix}, \quad j^{31} = -\frac{i}{2}\begin{vmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{vmatrix}, \quad j^{12} = -\frac{i}{2}\begin{vmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{vmatrix}.$$

To perform the non-relativistic approximation, we will apply the general method of projective operators:

$$P_+ = \frac{I + \gamma^0}{2} = \begin{vmatrix} I & 0 \\ 0 & 0 \end{vmatrix}, \qquad P_- = \frac{I - \gamma^0}{2} = \begin{vmatrix} 0 & 0 \\ 0 & I \end{vmatrix},$$

$$\Psi = \begin{vmatrix} \varphi_+(x) \\ \varphi_-(x) \end{vmatrix}, \quad \Psi_1 = P_1 \Psi = \begin{vmatrix} \varphi_+(x) \\ 0 \end{vmatrix}, \quad \Psi_2 = P_2 \Psi = \begin{vmatrix} 0 \\ \varphi_-(x) \end{vmatrix}.$$

The, the starting Dirac equation may be presented in the block form

$$\begin{vmatrix} i(\partial_0 + A_0) & i\hat{D}_k \sigma_k \\ -i\hat{D}_k \sigma_k & -i(\partial_0 + A_0) \end{vmatrix} \begin{vmatrix} \varphi_+(x) \\ \varphi_-(x) \end{vmatrix} = M \begin{vmatrix} \varphi_+(x) \\ \varphi_-(x) \end{vmatrix}, \tag{4}$$

where

$$\hat{D}_1 = \frac{1}{h_1}(\partial_1 + ieA_1) - i\frac{1}{2}G_{11}\sigma_1 - i\frac{1}{2}G_{21}\sigma_2 - i\frac{1}{2}G_{31}\sigma_3 = \hat{d}_1 - i\frac{1}{2}G_{11}\sigma_1 - i\frac{1}{2}G_{21}\sigma_2 - i\frac{1}{2}G_{31}\sigma_3,$$

$$\hat{D}_2 = \frac{1}{h_2}(\partial_2 + ieA_2) - \frac{1}{2}G_{12}\sigma_1 - i\frac{1}{2}G_{22}\sigma_2 - i\frac{1}{2}G_{32}\sigma_3 = \hat{d}_2 - -\frac{1}{2}G_{12}\sigma_1 - i\frac{1}{2}G_{22}\sigma_2 - i\frac{1}{2}G_{32}\sigma_3,$$

$$\hat{D}_3 = \frac{1}{h_3}(\partial_3 + ieA_3) - i\frac{1}{2}G_{13}\sigma_1 - i\frac{1}{2}G_{23}\sigma_2 - i\frac{1}{2}G_{33}\sigma_3 = \hat{d}_3 - i\frac{1}{2}G_{13}\sigma_1 - i\frac{1}{2}G_{23}\sigma_2 - i\frac{1}{2}G_{33}\sigma_3. \tag{5}$$

From (4), it followstwo blocks equations ($k = 1, 2, 3$)

$$iD_0\varphi_+ + i\hat{D}_k\sigma_k\varphi_- = M\varphi_+, \quad -i\hat{D}_k\sigma_k\varphi_+ - iD_0\varphi_- = M\varphi_-. \qquad (6)$$

According to general method, we are to separate the rest energy by the formal change $D_0 \Rightarrow (D_0 - iM)$; so obtain

$$i(D_0 - iM)\varphi_+ + i\hat{D}_k\sigma_k\varphi_- = M\varphi_+, \quad -i\hat{D}_k\sigma_k\varphi_+ - i(D_0 - iM)\varphi_- = M\varphi_-, \qquad (7)$$

whence after simplifying we get

$$i\frac{1}{M}D_0\varphi_+ + i\frac{1}{M}\hat{D}_k\sigma_k\varphi_- = 0, \quad -i\frac{1}{M}\hat{D}_k\sigma_k\varphi_+ - i\frac{1}{M}D_0\varphi_- = 2\varphi_-. \qquad (8)$$

It is known that we should assume the following orders of smallness for involved quantities:

$$\varphi_+ : 1, \quad \varphi_- : x, \quad \frac{1}{M}\hat{D}_j : x, \quad \frac{1}{M}G_{kl} : x, \quad \frac{1}{M}D_0 : x^2, \qquad (9)$$

Taking this into account, we derive two equations of order x and order x^2:

$$\varphi_- = -i\frac{1}{2M}\hat{D}_n\sigma_n\varphi_+, \quad i\frac{1}{M}D_0\varphi_+ + i\frac{1}{M}\hat{D}_k\sigma_k\varphi_- = 0. \qquad (10)$$

Eliminationg the small component φ_-, we arrive at the following equation for the large component φ_+:

$$iD_0\varphi_+ = -\frac{1}{2M}(\hat{D}_1\sigma_1 + \hat{D}_2\sigma_2 + \hat{D}_3\sigma_3)(\hat{D}_1\sigma_1 + \hat{D}_2\sigma_2 + \hat{D}_3\sigma_3)\varphi_+. \qquad (11)$$

Let us detail this equation (note the new notation $\varphi_+ = \Psi$):

$$iD_0\Psi = -\frac{1}{2M}(\hat{D}_1\sigma_1 + \hat{D}_2\sigma_2 + \hat{D}_3\sigma_3)(\hat{D}_1\sigma_1 + \hat{D}_2\sigma_2 + \hat{D}_3\sigma_3)\Psi; \qquad (12)$$

Taking into account identities

$$\hat{D}_1\sigma_1 = \hat{d}_1\sigma_1 - i\frac{1}{2}G_{11}\sigma_1\sigma_1 - i\frac{1}{2}G_{21}\sigma_2\sigma_1 - i\frac{1}{2}G_{31}\sigma_3\sigma_1,$$

$$\hat{D}_2\sigma_2 = \hat{d}_2\sigma_2 - i\frac{1}{2}G_{12}\sigma_1\sigma_2 - i\frac{1}{2}G_{22}\sigma_2\sigma_2 - i\frac{1}{2}G_{32}\sigma_3\sigma_2,$$

$$\hat{D}_3\sigma_3 = \hat{d}_3\sigma_3 - i\frac{1}{2}G_{13}\sigma_1\sigma_3 - i\frac{1}{2}G_{23}\sigma_2\sigma_3 - i\frac{1}{2}G_{33}\sigma_3\sigma_3.$$

and the formulas for products of the Pauli matrices, we obtain

$$\hat{D}_1\sigma_1 = \hat{d}_1\sigma_1 - i\frac{1}{2}G_{11} - \frac{1}{2}G_{21}\sigma_3 + \frac{1}{2}G_{31}\sigma_2,$$
$$\hat{D}_2\sigma_2 = \hat{d}_2\sigma_2 + \frac{1}{2}G_{12}\sigma_3 - i\frac{1}{2}G_{22} - \frac{1}{2}G_{32}\sigma_1, \qquad (13)$$
$$\hat{D}_3\sigma_3 = \hat{d}_3\sigma_3 - \frac{1}{2}G_{13}\sigma_2 + \frac{1}{2}G_{23}\sigma_1 - i\frac{1}{2}G_{33}.$$

Thus, we have the identity

$$\hat{D}_1\sigma_1 + \hat{D}_2\sigma_2 + \hat{D}_3\sigma_3 = (\hat{d}_1\sigma_1 + \hat{d}_2\sigma_2 + \hat{d}_3\sigma_3) -$$

$$-i\frac{1}{2}(G_{11}+G_{22}+G_{33})+\frac{1}{2}(G_{23}-G_{32})\sigma_1+\frac{1}{2}(G_{31}-G_{13})\sigma_2+\frac{1}{2}(G_{12}-G_{21})\sigma_3.$$

It may be presented shorter (for static case, we have $B\equiv 0$)

$$\hat{D}_1\sigma_1+\hat{D}_2\sigma_2+\hat{D}_3\sigma_3=(\hat{d}_1\sigma_1+\hat{d}_2\sigma_2+\hat{d}_3\sigma_3)-iB+B_1\sigma_1+B_2\sigma_2+B_3\sigma_3. \qquad (14)$$

Therefore, the Pauli equation in curvilinear coordinates takes on the form

$$iD_0\Psi=-\frac{1}{2M}(-iB+(\hat{d}_1+B_1)\sigma_1+(\hat{d}_2+B_2)\sigma_2+(\hat{d}_3+B_3)\sigma_3))\times$$
$$\times(-iB+(\hat{d}_1+B_1)\sigma_1+(\hat{d}_2+B_2)\sigma_2+(\hat{d}_3+B_3)\sigma_3))\Psi. \qquad (15)$$

The detailed structure of the Pauli equation

Let us detail the product of tho multipliers

$$\Pi^2=(-iB+(\hat{d}_1+B_1)\sigma_1+(\hat{d}_2+B_2)\sigma_2+(\hat{d}_3+B_3)\sigma_3))\times$$
$$\times(-iB+(\hat{d}_1+B_1)\sigma_1+(\hat{d}_2+B_2)\sigma_2+(\hat{d}_3+B_3)\sigma_3))=$$
$$=-iB(-iB+(\hat{d}_1+B_1)\sigma_1+(\hat{d}_2+B_2)\sigma_2+(\hat{d}_3+B_3)\sigma_3))+$$
$$+(\hat{d}_1+B_1)(-iB\sigma_1+(\hat{d}_1+B_1)+i(\hat{d}_2+B_2)\sigma_3-i(\hat{d}_3+B_3)\sigma_2))+$$
$$+(\hat{d}_2+B_2)(-iB\sigma_2-i(\hat{d}_1+B_1)\sigma_3+(\hat{d}_2+B_2)+i(\hat{d}_3+B_3)\sigma_1))+$$
$$+(\hat{d}_3+B_3)(-iB\sigma_3+i(\hat{d}_1+B_1)\sigma_2-i(\hat{d}_2+B_2)\sigma_1+(\hat{d}_3+B_3))).$$

Temporally, we will apply the shortening notations

$$\hat{d}_k+B_k=\frac{1}{h_k}(\partial_k+ieA_k)+B_k=Q_k,\, k=1,2,3\,;$$

then we obtain

$$\Pi^2=-iB(-iB+Q_1\sigma_1+Q_2\sigma_2+Q_3\sigma_3)+Q_1(-iB\sigma_1+Q_1+iQ_2\sigma_3-iQ_3\sigma_2)+$$
$$+Q_2(-iB\sigma_2-iQ_1\sigma_3+Q_2+iQ_3\sigma_1)+Q_3(-iB\sigma_3+iQ_1\sigma_2-iQ_2\sigma_1+Q_3),$$

that is

$$\Pi^2=-B^2-iBQ_1\sigma_1-iBQ_2\sigma_2-iBQ_3\sigma_3-iQ_1B\sigma_1+Q_1Q_1+iQ_1Q_2\sigma_3-iQ_1Q_3\sigma_2-$$
$$-iQ_2B\sigma_2-iQ_2Q_1\sigma_3+Q_2Q_2+iQ_2Q_3\sigma_1-iQ_3B\sigma_3+iQ_3Q_1\sigma_2-iQ_3Q_2\sigma_1+Q_3Q_3,$$

or

$$\Pi^2=+(Q_1Q_1+Q_2Q_2+Q_3Q_3)-B^2-$$
$$-iB(Q_1\sigma_1+Q_2\sigma_2+Q_3\sigma_3)-i(Q_1B\sigma_1+Q_2B\sigma_2+Q_3B\sigma_3)+$$
$$+i(Q_1Q_2-Q_2Q_1)\sigma_3+i(Q_3Q_1-Q_1Q_3)\sigma_2+i(Q_2Q_3-Q_3Q_2)\sigma_1;$$

for generality we will preserve the term B relevant for non-static metrics.

Let us turn to the structure of the Pauli equation

$$iD_0\Psi = -\frac{1}{2M}\Pi^2 = -\frac{1}{2M}\{(Q_1Q_1 + Q_2Q_2 + Q_3Q_3) - B^2 -$$
$$-i(BQ_1 + Q_1B)\sigma_1 - i(BQ_2 + Q_2B)\sigma_2 - i(BQ_3 + Q_3B)\sigma_3) +$$
$$+i(Q_1Q_2 - Q_2Q_1)\sigma_3 + i(Q_3Q_1 - Q_1Q_3)\sigma_2 + i(Q_2Q_3 - Q_3Q_2)\sigma_1\}\Psi. \qquad (16)$$

We should find the commutators $[Q_k, Q_l]_-$:

$$(Q_2Q_3 - Q_3Q_2)\Psi = \{((\frac{1}{h_2}\partial_2\frac{1}{h_3})\partial_3 - (\frac{1}{h_3}\partial_3\frac{1}{h_2})\partial_2) +$$
$$+ie((\frac{1}{h_2}\partial_2\frac{1}{h_3})A_3 - (\frac{1}{h_3}\partial_3\frac{1}{h_2})A_2) + ie\frac{1}{h_2}\frac{1}{h_3}(\partial_2A_3 - \partial_3A_2) + (\frac{1}{h_2}\partial_2B_3 - \frac{1}{h_3}\partial_3B_2)\}\Psi;$$

$$(Q_3Q_1 - Q_1Q_3)\Psi = \{((\frac{1}{h_3}\partial_3\frac{1}{h_1})\partial_1 - (\frac{1}{h_1}\partial_1\frac{1}{h_3})\partial_3) +$$
$$+ie((\frac{1}{h_3}\partial_3\frac{1}{h_1})A_1 - (\frac{1}{h_1}\partial_1\frac{1}{h_3})A_3) + ie\frac{1}{h_3}\frac{1}{h_1}(\partial_3A_1 - \partial_1A_3) + (\frac{1}{h_3}\partial_3B_1 - \frac{1}{h_1}\partial_1B_3)\}\Psi;$$

$$(Q_1Q_2 - Q_2Q_1)\Psi = \{((\frac{1}{h_1}\partial_1\frac{1}{h_2})\partial_2 - (\frac{1}{h_2}\partial_2\frac{1}{h_1})\partial_1) +$$
$$+ie((\frac{1}{h_1}\partial_1\frac{1}{h_2})A_2 - (\frac{1}{h_2}\partial_2\frac{1}{h_1})A_1) + ie\frac{1}{h_1}\frac{1}{h_2}(\partial_1A_2 - \partial_2A_1) + (\frac{1}{h_1}\partial_1B_2 - \frac{1}{h_2}\partial_2B_1)\}\Psi.$$

With the special notations

$$\frac{1}{h_1}\partial_1 = \partial_{(1)}, \quad \frac{1}{h_2}\partial_2 = \partial_{(2)}, \quad \frac{1}{h_3}\partial_3 = \partial_{(3)},$$
$$\partial_2A_3 - \partial_3A_2 = F_{23}, \quad \partial_3A_1 - \partial_1A_3 = F_{31}, \quad \partial_1A_2 - \partial_2A_1 = F_{12},$$

the above formulas read more symmetrically

$$(Q_2Q_3 - Q_3Q_2)\Psi = \{((\partial_{(2)}\frac{1}{h_3})\partial_3 - (\partial_{(3)}\frac{1}{h_2})\partial_2) +$$
$$+ie((\partial_{(2)}\frac{1}{h_3})A_3 - (\partial_{(3)}\frac{1}{h_2})A_2) + ie\frac{1}{h_2}\frac{1}{h_3}F_{23} + (\partial_{(2)}B_3 - \partial_{(3)}B_2)\}\Psi;$$

$$(Q_3Q_1 - Q_1Q_3)\Psi = \{((\partial_{(3)}\frac{1}{h_1})\partial_1 - (\partial_{(1)}\frac{1}{h_3})\partial_3) +$$
$$+ie((\partial_{(3)}\frac{1}{h_1})A_1 - (\partial_{(1)}\frac{1}{h_3})A_3) + ie\frac{1}{h_3}\frac{1}{h_1}F_{31} + (\partial_{(3)}B_1 - \partial_{(1)}B_3)\}\Psi;$$

$$(Q_1Q_2 - Q_2Q_1)\Psi = \{((\partial_{(1)}\frac{1}{h_2})\partial_2 - (\partial_{(2)}\frac{1}{h_1})\partial_1) +$$

$$+ie((\partial_{(1)}\frac{1}{h_2})A_2-(\partial_{(2)}\frac{1}{h_1})A_1)+ie\frac{1}{h_1}\frac{1}{h_2}F_{12}+(\partial_{(1)}B_2-\partial_{(2)}B_1)\}\Psi.$$

They may be presented yet shorter

$$(Q_2Q_3-Q_3Q_2)\Psi=\{((\partial_{(2)}\frac{1}{h_3})(\partial_3+ieA_3)-(\partial_{(3)}\frac{1}{h_2})(\partial_2+ieA_2))+$$

$$+ie\frac{1}{h_2}\frac{1}{h_3}F_{23}+(\partial_{(2)}B_3-\partial_{(3)}B_2)\}\Psi;$$

$$(Q_3Q_1-Q_1Q_3)\Psi=\{((\partial_{(3)}\frac{1}{h_1})(\partial_1+ieA_1)-(\partial_{(1)}\frac{1}{h_3})(\partial_3+ieA_3))+$$

$$+ie\frac{1}{h_3}\frac{1}{h_1}F_{31}+(\partial_{(3)}B_1-\partial_{(1)}B_3)\}\Psi;$$

$$(Q_1Q_2-Q_2Q_1)\Psi=\{((\partial_{(1)}\frac{1}{h_2})(\partial_2+ieA_2))-(\partial_{(2)}\frac{1}{h_1})(\partial_1+ieA_1))+$$

$$+ie\frac{1}{h_1}\frac{1}{h_2}F_{12}+(\partial_{(1)}B_2-\partial_{(2)}B_1)\}\Psi.$$

For non-static metrics, we would add expressions for three terms

$$-i(BQ_1+Q_1B);\quad -i(BQ_2+Q_2B);\quad -i(BQ_3+Q_3B));$$

we omit them.

For static metrics, the Pauli equation has the following general structure

$$iD_0\Psi=-\frac{1}{2M}\Pi^2=-\frac{1}{2M}\{(Q_1Q_1+Q_2Q_2+Q_3Q_3)+$$

$$+i(Q_2Q_3-Q_3Q_2)\sigma_1+i(Q_3Q_1-Q_1Q_3)\sigma_2++i(Q_1Q_2-Q_2Q_1)\sigma_3\}\Psi. \qquad (17)$$

Pauli equation in the first order form

Let us turn to the non-relativistic equation (12)

$$iD_0\Psi=-\frac{1}{2M}(\hat{D}_1\sigma_1+\hat{D}_2\sigma_2+\hat{D}_3\sigma_3)(\hat{D}_1\sigma_1+\hat{D}_2\sigma_2+\hat{D}_3\sigma_3)\Psi, \qquad (18)$$

and give other but equivalent formulation of this problem, which is based on the first order differential equations. To this end, ltutning to the operator-multiplier entering eq. (18):

$$D=(\hat{D}_1\sigma_1+\hat{D}_2\sigma_2+\hat{D}_3\sigma_3)=-iB+(\hat{d}_1+B_1)\sigma_1+(\hat{d}_2+B_2)\sigma_2+(\hat{d}_3+B_3)\sigma_3;$$

We fix its matrix structure

$$D=\begin{vmatrix}(\hat{d}_3-iB+B_3) & (\hat{d}_1-i\hat{d}_2+B_1-iB_2)\\(\hat{d}_1+i\hat{d}_2+B_1+iB_2) & (-\hat{d}_3-iB-B_3)\end{vmatrix}. \qquad (19)$$

As the next step, let us introduce a subsidiary 2-component function Φ:

$$\Psi = \begin{vmatrix} \Psi_1 \\ \Psi_2 \end{vmatrix}, \quad \Phi = \begin{vmatrix} \Phi_1 \\ \Phi_2 \end{vmatrix} = D\Psi = 1, \tag{20}$$

so that

$$\Phi_1 = (\hat{d}_3 - iB + B_3)\Psi_1 + (\hat{d}_1 - i\hat{d}_2 + B_1 - iB_2)\Psi_2,$$

$$\Phi_2 = (-\hat{d}_3 - iB - B_3)\Psi_2 + (\hat{d}_1 + i\hat{d}_2 + B_1 + iB_2)\Psi_1.$$

In this way, instead of the above Pauli equation we get equivalent system of four first order equations

$$\Phi_1 = (+\hat{d}_3 - iB + B_3)\Psi_1 + (\hat{d}_1 - i\hat{d}_2 + B_1 - iB_2)\Psi_2,$$

$$\Phi_2 = (-\hat{d}_3 - iB - B_3)\Psi_2 + (\hat{d}_1 + i\hat{d}_2 + B_1 + iB_2)\Psi_1, \tag{21}$$

$$2iMD_0\Psi_1 + (+\hat{d}_3 - iB + B_3)\Phi_1 + (\hat{d}_1 - i\hat{d}_2 + B_1 - iB_2)\Phi_2 = 0,$$

$$2iMD_0\Psi_2 + (-\hat{d}_3 - iB - B_3)\Phi_2 + (\hat{d}_1 + i\hat{d}_2 + B_1 + iB_2)\Phi_1 = 0, \tag{22}$$

Where

$$\hat{d}_1 = \frac{1}{h_1}\partial_1 + ieA_1, \quad \hat{d}_2 = \frac{1}{h_2}\partial_2 + ieA_2, \quad \hat{d}_3 = \frac{1}{h_3}\partial_3 + ieA_3;$$

$$B_1 = \frac{1}{2}(G_{23} - G_{32}), \quad B_2 = \frac{1}{2}(G_{31} - G_{13}), \quad B_3 = \frac{1}{2}(G_{12} - G_{21}), \quad B = \frac{1}{2}(G_{11} + G_{22} + G_{33}).$$

In static metrics, equations become simpler

$$\Phi_1 = (+\hat{d}_3 + B_3)\Psi_1 + (\hat{d}_1 - i\hat{d}_2 + B_1 - iB_2)\Psi_2,$$

$$\Phi_2 = (-\hat{d}_3 - B_3)\Psi_2 + (\hat{d}_1 + i\hat{d}_2 + B_1 + iB_2)\Psi_1; \tag{23}$$

$$2iMD_0\Psi_1 + (+\hat{d}_3 + B_3)\Phi_1 + (\hat{d}_1 - i\hat{d}_2 + B_1 - iB_2)\Phi_2 = 0,$$

$$2iMD_0\Psi_2 + (-\hat{d}_3 - B_3)\Phi_2 + (\hat{d}_1 + i\hat{d}_2 + B_1 + iB_2)\Phi_1 = 0. \tag{24}$$

The usual 2-component Pauli equation is obtained by eliminating auxiliary variables Φ_1, Φ_2.

Particle in magnetic field, cylindrical coordinates

Let us write down the needed initial definitions

$$(r, \phi, z), \quad G_{32} = +\frac{1}{2}\frac{1}{r}, \quad B_1 = -\frac{1}{2}\frac{1}{r}, \qquad \hat{d}_1 = \partial_r, \quad \hat{d}_2 = \frac{1}{r}(\partial_\phi - ie\frac{Br^2}{2}), \quad \hat{d}_3 = \partial_z;$$

the system of Pauli first order equations reads

$$\Phi_1 = +\partial_z\Psi_1 + (\partial_r - \frac{i}{r}(\partial_\phi - ie\frac{Br^2}{2}) - \frac{1/2}{r})\Psi_2,$$

$$\Phi_2 = -\partial_z\Psi_2 + (\partial_r + \frac{i}{r}(\partial_\phi - ie\frac{Br^2}{2}) - \frac{1/2}{r})\Psi_1,$$

$$2iM\partial_t\Psi_1+\partial_z\Phi_1+(\partial_r-\frac{i}{r}(\partial_\phi-ie\frac{Br^2}{2})-\frac{1/2}{r})\Phi_2=0\,,$$
$$2iM\partial_t\Psi_2-\partial_z\Phi_2+(\partial_r+\frac{i}{r}(\partial_\phi-ie\frac{Br^2}{2})-\frac{1/2}{r})\Phi_1=0\,.$$

The general substitution for the wave function should be

$$\Psi=e^{-iEt}e^{im\phi}e^{ikz}\begin{vmatrix}f_1(r)\\f_2(r)\end{vmatrix},\quad \Phi=e^{-iEt}e^{im\phi}e^{ikz}\begin{vmatrix}g_1(r)\\g_2(r)\end{vmatrix};\qquad(25)$$

correspondingly we get more simple system

$$\begin{aligned}&g_1=+ikf_1+(\frac{d}{dr}+\frac{1}{r}(m-e\frac{Br^2}{2})-\frac{1/2}{r})f_2,\\&g_2=-ikf_2+(\frac{d}{dr}-\frac{1}{r}(m-e\frac{Br^2}{2})-\frac{1/2}{r})f_1,\\&2MEf_1+ikg_1+(\frac{d}{dr}+\frac{1}{r}(m-e\frac{Br^2}{2})-\frac{1/2}{r})g_2=0,\\&2MEf_2-ikg_2+(\frac{d}{dr}-\frac{1}{r}(m-e\frac{Br^2}{2})-\frac{1/2}{r})g_1=0.\end{aligned}\qquad(26)$$

Let us introduce the shortening notations (also for brevity let $B\Rightarrow B$)

$$\hat{a}_m=\frac{d}{dr}+\frac{1}{r}(m-1/2-\frac{Br^2}{2}),\quad \hat{b}_m=\frac{d}{dr}-\frac{1}{r}(m+1/2-\frac{Br^2}{2});\qquad(27)$$

then the equations read

$$\begin{aligned}&g_1=+ikf_1+\hat{a}_m f_2,\quad g_2=-ikf_2+\hat{b}_m f_1,\\&2MEf_1+ikg_1+\hat{a}_m g_2=0,\quad 2MEf_2-ikg_2+\hat{b}_m g_1=0.\end{aligned}\qquad(28)$$

Eliminating the redundant variables, we derive two second order equations

$$2ME\,f_1-k^2f_1+ik\hat{a}_m f_2-ik\hat{a}_m f_2+\hat{a}_m\hat{b}_m f_1=0,$$
$$2ME\,f_2-k^2f_2-ik\hat{b}_m f_1+ik\hat{b}_m f_1+\hat{b}_m\hat{a}_m f_1=0;$$

so we arrive at two separated equations

$$\begin{aligned}&(\hat{a}_m\hat{b}_m+2ME-k^2)\,f_1(r)=0,\\&(\hat{b}_m\hat{a}_m+2ME-k^2)\,f_2(r)=0.\end{aligned}\qquad(29)$$

They are readily solvable, and then lead to the known Landau levels (see for instance in [5]). One can use the following two independent solutions

$$\Psi_1=e^{-iEt}e^{im\phi}e^{ikz}\begin{vmatrix}f_1(r)\\0\end{vmatrix},\quad \Psi_2=e^{-iEt}e^{im\phi}e^{ikz}\begin{vmatrix}0\\f_2(r)\end{vmatrix}.\qquad(30)$$

We can elaborate another way for studying the problem, more preferable for further extension. To this end, we turn back the the first order system (34)

$$\begin{aligned} &g_1 = +ikf_1 + \hat{a}_m f_2, \\ &g_2 = -ikf_2 + \hat{b}_m f_1, \\ &2MEf_1 + ikg_1 + \hat{a}_m g_2 = 0, \\ &2MEf_2 - ikg_2 + \hat{b}_m g_1 = 0; \end{aligned} \quad (31)$$

evidently, for existence of solutions, we should assume the algebraic constraints

$$g_1 = \mu_1 f_1 \qquad g_2 = \mu_2 f_2; \quad (32)$$

this leads to

$$\begin{aligned} &\mu_1 f_1 = ikf_1 + \hat{a}_m f_2, \\ &\mu_2 f_2 = -ikf_2 + \hat{b}_m f_1, \\ &2MEf_1 + ik\mu_1 f_1 + \hat{a}_m \mu_2 f_2 = 0, \\ &2MEf_2 - ik\mu_2 f_2 + \hat{b}_m \mu_1 f_1 = 0; \end{aligned} \quad (33)$$

and then we should assume the differential constraints

$$\hat{a}_m f_2 = C_1 f_1, \quad \hat{b}_m f_1 = C_2 f_2;$$

then the system reads

$$\begin{aligned} &\mu_1 f_1 = ikf_1 + C_1 f_1 \quad \Rightarrow \quad \mu_1 = ik + C_1, \\ &\mu_2 f_2 = -ikf_2 + C_2 f_2 \quad \Rightarrow \quad \mu_2 = -ik + C_2, \\ &2MEf_1 + ik\mu_1 f_1 + \mu_2 C_1 f_1 = 0 \quad \Rightarrow \quad 2ME + ik\mu_1 + \mu_2 C_1 = 0, \\ &2MEf_2 - ik\mu_2 f_2 + \mu_1 C_2 f_2 = 0 \quad \Rightarrow \quad 2ME - ik\mu_2 + \mu_1 C_2 = 0. \end{aligned} \quad (34)$$

This algebraic system can be readily solved

$$\begin{aligned} &\mu_1 = ik + C, \quad \mu_2 = -ik + C; \\ &2ME + ik\mu_1 + \mu_2 C = 0, \quad 2ME - ik\mu_2 + \mu_1 C = 0, \end{aligned} \quad (35)$$

that

$$\begin{aligned} &2ME + ik(ik + C) + (-ik + C)C = 0, \\ &2ME - ik(-ik + C) + (ik + C)C = 0; \end{aligned}$$

whence it follows

$$C = \pm i\sqrt{2ME - k^2}, \quad \mu_1 = ik \pm i\sqrt{2ME - k^2}, \quad \mu_2 = -ik \pm i\sqrt{2ME - k^2}; \quad (36)$$

μ_1, μ_2 determine the constraints in the first order system

$$g_1 = \mu_1 f_1, g_2 = \mu_2 f_2, \quad C = \pm i\sqrt{2ME - k^2}. \quad C^2 = -(2ME - k^2).$$

In turn, the differential constraints gives separate equations (without lost of generality one can set $C_1 = C_2 = C$)

$$(\hat{a}_m \hat{b}_m - C^2) f_1(r) = 0, \qquad (\hat{b}_m \hat{a}_m - C^2) f_2(r) = 0; \quad (37)$$

They coincide with equatiopns (29). We should remember that the functions $f_1(r),\ f_2(r)$ are linked to each other by the first order differential constraints

$$(\hat{a}_m\hat{b}_m - C^2)\, f_1(r) = 0, \quad f_2(r) = \frac{1}{C}\hat{b}_m f_1(r),$$

$$(\hat{b}_m\hat{a}_m - C^2)\, f_2(r) = 0, \quad f_1(t) = \frac{1}{C}\hat{a}_m f_2(r);$$

So we can introduce two new solutions

$$\Psi_+ = \begin{vmatrix} f_1 \\ \frac{1}{C_+}\hat{b}_m f_1 \end{vmatrix}, \quad \Psi_- = \begin{vmatrix} f_1 \\ \frac{1}{C_-}\hat{b}_m f_1 \end{vmatrix}; \qquad (38)$$

we readily find relationships between the above solutions Ψ_1, Ψ_2 (30) and the new ones Ψ_+, Ψ_-:

$$\Psi_+ - \Psi_- = (\frac{1}{C_+} - \frac{1}{C_-})\Psi_2, \qquad C_+\Psi_+ - C_-\Psi_- = (C_+ - C_-)\Psi_1. \qquad (39)$$

Conclusions

The non-relativistic Pauli approximation for a spin ½ particle was studied in arbitatry curvilinear orthogonal coordinates.The final goal is to develop a unified approach to separating the variables in equation for spin ½ particle in all 12 systems of orthogonal coordinates in the flat space, first in more simple Pauli approximation. We derive general structure for covariant Pauli equation in arbitrary orthogonal coordinates, it includes the relevant tetrads and Ricci rotation coefficients. Because the possibility to develop a unified approach to separating the variables in ordinary presentation of the covariant Pauli equation is rather problematitic, other way for studying the problem is proposed. It is based on transforming the usual 2-component Pauli equation to an equaivalent system of first order four differential equations, by introducing two auxiliary components, because it is known that the task of separating the variables in the system of differential equations in partial derivatives may be solved easier for the first order systems. One simple example for illustating this approach, is given; a spin ½ particle problem in presence of external magnetic field when using the cylindrical coordinates.

The more perspective for generalization is seems to be the method that is based on first order form of Pauli theory.

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