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Calb – Ramond Field, Solutions with Spherical Symmetry, the Gauge Degrees of Freedom

In the present paper, the system of 10 equations for Calb – Ramond particle is studied in spherical coordinates. For this particle, in contrast to Maxwell theory, the antisymmetric tensor represents gauge variables, and 4-vector relates to physically observable ones. After separating the variables we get the first order system of 10 radial equations. By diagonalysing the space reflection operator, we get to more simple subsystems of 4 and 6 equations, related to states with parities $P=(-1)^{j+1}$ and $P=(-1)^{j}$ respectively. For parity $P=(-1)^{j+1}$ the system of 4 equations has two independent solutions, they both describe two gauge states. For parity $P=(-1)^{j}$, the system of 6 equations has 2 independent solutions; one of them is purely gauge, and the other includes both observable and gauge variables. Therefore, for Calb – Ramond particle there exist only one physically observable state with spherical symmetry, and three states are gauge ones. Recall that in Maxwell theory, exist 2 physically observable states, and 2 pure gauge states.

Key words: *Calb – Ramond field, spherical symmetry, separation of the variables, exact solutions, the gauge degrees of freedom.*

ПОЛЕ КАЛЬБА – РАМОНА, РЕШЕНИЯ СО СФЕРИЧЕСКОЙ СИММЕТРИЕЙ, КАЛИБРОВОЧНЫЕ СТЕПЕНИ СВОБОДЫ

Исследуется система из 10 уравнений для безмассовой частицы Кальба – Рамона в сферических координатах. Для этой частицы в отличие от теории Максвелла антисимметричный тензор представляет калибровочные переменные, а 4-вектор соответствует наблюдаемым величинам. После разделения переменных получена система из 10 радиальных уравнений. С помощью диагонализации оператора пространственного отражения получаем более простые подсистемы из 4 и 6 уравнений, относящиеся к состояниям с четностью $P=(-1)^{j+1}$ и $P=(-1)^{j}$ соответственно. Для четности $P=(-1)^{j+1}$ система из 4 уравнений имеет два независимых решения, оба чисто калибровочные. Для четности $P=(-1)^{j}$ система из 6 уравнений имеет два независимых решения; одно чисто калибровочное, второе включает наблюдаемые и калибровочные переменные. Следовательно, для частицы Кальба – Рамона существует только одно физическое наблюдаемое состояние, три остальных чисто калибровочные. В теории Максвелла существуют два физических и 2 калибровочных решения.

Ключевые слова: *поле Кальба – Рамона, сферическая симметрия, разделение переменных, точные решения, калибровочные степени свободы.*

Introduction

The field of Ogievetsky – Polubarinov [1], now it is mostly called as Calb – Ramond field [2], takes much attention in the current literature; for instance see in [3; 4]. In this theory, in contrast to Maxwell theory, the antisymmetric tensor represents the gauge variables, and

4-vector relates to physically observable ones. In the present paper, we focus on the problem of finding in explicit form all independent solutions for this field with spherical symmetry.

1. Exact solutions

We start with the known radial equations [5] for massive Stueckelberg field [6; 7], restricted to the ordinary spin 1 particle:

$$-E_2' - \frac{2}{r}E_2 - \frac{1}{r}\frac{a}{\sqrt{2}}(E_1 + E_3) = Mh_0, \quad i\epsilon E_1 - B_3' - \frac{1}{r}B_3 + \frac{1}{r}\frac{a}{\sqrt{2}}B_2 = Mh_1,$$

$$i\epsilon E_2 - \frac{1}{r}\frac{a}{\sqrt{2}}(B_1 - B_3) = Mh_2, \quad i\epsilon E_3 + B_1' + \frac{1}{r}B_1 - \frac{1}{r}\frac{a}{\sqrt{2}}B_2 = Mh_3,$$

$$-i\epsilon h_1 + \frac{1}{r}\frac{a}{\sqrt{2}}h_0 = ME_1, \quad -i\epsilon h_2 - h_0' = ME_2,$$

$$-i\epsilon h_3 + \frac{1}{r}\frac{a}{\sqrt{2}}h_0 = ME_3, \quad h_3' + \frac{1}{r}h_3 + \frac{1}{r}\frac{a}{\sqrt{2}}h_2 = MB_1,$$

$$-\frac{1}{r}\frac{a}{\sqrt{2}}h_1 + \frac{1}{r}\frac{a}{\sqrt{2}}h_3 = MB_2, \quad -h_1' - \frac{1}{r}h_1 - \frac{1}{r}\frac{a}{\sqrt{2}}h_2 = MB_3.$$

The variables h_a refer to 4-vector, and E_i, B_i refer to antisymmetric tensor. In massless Calb – Ramond case, this system becomes slightly different

$$-E_2' - \frac{2}{r}E_2 - \frac{1}{r}\frac{a}{\sqrt{2}}(E_1 + E_3) = h_0, \quad i\epsilon E_1 - B_3' - \frac{1}{r}B_3 + \frac{1}{r}\frac{a}{\sqrt{2}}B_2 = h_1,$$

$$i\epsilon E_2 - \frac{1}{r}\frac{a}{\sqrt{2}}(B_1 - B_3) = h_2, \quad i\epsilon E_3 + B_1' + \frac{1}{r}B_1 - \frac{1}{r}\frac{a}{\sqrt{2}}B_2 = h_3,$$

$$-i\epsilon h_1 + \frac{1}{r}\frac{a}{\sqrt{2}}h_0 = 0, \quad -i\epsilon h_2 - h_0' = 0,$$

$$-i\epsilon h_3 + \frac{1}{r}\frac{a}{\sqrt{2}}h_0 = 0, \quad h_3' + \frac{1}{r}h_3 + \frac{1}{r}\frac{a}{\sqrt{2}}h_2 = 0,$$

$$-\frac{1}{r}\frac{a}{\sqrt{2}}h_1 + \frac{1}{r}\frac{a}{\sqrt{2}}h_3 = 0, \quad -h_1' - \frac{1}{r}h_1 - \frac{1}{r}\frac{a}{\sqrt{2}}h_2 = 0.$$

These equations substantially differ from the corresponding equations for Maxwell theory.

There exists possibility to diagonalize the space reflection operator, this results in two types of restrictions depending on the parity:

$$P = (-1)^{j+1}, \quad h_0 = 0, h_2 = 0, h_3 = -h_1, E_3 = -E_1, E_2 = 0, B_3 = +B_1;$$

$$P = (-1)^{j}, \quad h_3 = +h_1, B_3 = -B_1, B_2 = 0, E_3 = +E_1.$$

Taking them into account in the radial system, we obtain two subsystems of 4 and 6 equations.

Let $P = (-1)^{j+1}$, then we have 4 equations

$$i\epsilon E_1 - B_1' - \frac{B_1}{r} + \frac{a}{\sqrt{2}}\frac{B_2}{r} = h_1, \quad -i\epsilon h_1 = 0, \quad -h_1' - \frac{h_1}{r} = 0, \quad -\frac{a\sqrt{2}h_1}{r} = 0;$$

note that vector variables vanish $h_0 = h_1 = h_2 = h_3 = 0$ identically; whereas 3 gauge components obey the equations

$$E_3 = -E_1, \quad E_2 = 0, \quad B_3 = +B_1, \quad i\epsilon E_1 = (\frac{d}{dr} + \frac{1}{r})B_1 - \frac{a}{\sqrt{2}r}B_2.$$

Here two independent solutions are possible:

$$\begin{aligned} &1. \quad B_2 = 0, B_3 = B_1 \textit{ is arbitrary}, E_3 = -E_1, i\epsilon E_1 = (\frac{d}{dr} + \frac{1}{r})B_1; \\ &2. \quad B_3 = B_1 = 0, B_2 \textit{ is arbitrary}, \quad E_3 = -E_1, E_2 = 0, i\epsilon E_1 = -\frac{a}{\sqrt{2}}B_2. \end{aligned} \tag{1}$$

It is evident, that in (1) we describe two gauge solutions, because each of them is determined by an arbitrary function (correspondingly by B_2 and B_1). Thus, for states with parity $P = (-1)^{j+1}$, any physically observable states do not exist.

For states with the opposite parity $P = (-1)^j$, we have 6 equations:

$$1) \quad -E_2' - \frac{2}{r}E_2 - \frac{a\sqrt{2}}{r}E_1 = h_0,$$

$$2) \quad h' + i\epsilon E_2 - \frac{a\sqrt{2}}{r}B_1 = h_2,$$

$$3) \quad -\frac{1}{2}\frac{a\sqrt{2}}{r}h + i\epsilon E_1 + B_1' + \frac{B_1}{r} = h_1;$$

$$4) \quad -i\epsilon h_2 - h_0' = 0,$$

$$5) \quad -i\epsilon h_1 + \frac{1}{2}\frac{a\sqrt{2}}{r}h_0 = 0,$$

$$6) \quad h_1' + \frac{h_1}{r} + \frac{1}{2}\frac{a\sqrt{2}}{r}h_2 = 0;$$

three last equations contain only variables related to the 4-vector.

Let us consider the first 3 equations, in which the gauge variables E_1, E_2, B_1 are linked to the variables h, h_0, h_1, h_2:

$$1) \quad -E_2' - \frac{2}{r}E_2 - \frac{a\sqrt{2}}{r}E_1 = h_0,$$

$$2) \quad E_2 = \frac{1}{i\epsilon}[h_2 + \frac{a\sqrt{2}}{r}B_1],$$

$$3) \quad E_1 = \frac{1}{i\epsilon}[h_1 - B_1' - \frac{B_1}{r}].$$

Let us substitute expressions for E_1 and E_2 into equation 1):

$$\frac{d}{dr}[h_2 + \frac{a\sqrt{2}}{r}B_1] + \frac{2}{r}[h_2 + \frac{a\sqrt{2}}{r}B_1] + \frac{a\sqrt{2}}{r}[h_1 - \frac{d}{dr}B_1 - \frac{B_1}{r}] = -i\epsilon h_0;$$

whence it follows

$$\frac{d}{dr}h_2 - \frac{a\sqrt{2}}{r^2}B_1 + \frac{a\sqrt{2}}{r}\frac{d}{dr}B_1 + \frac{2}{r}h_2 + \frac{2\sqrt{2}a}{r^2}B_1 + \frac{a\sqrt{2}}{r}h_1 - \frac{a\sqrt{2}}{r}\frac{d}{dr}B_1 - \frac{a\sqrt{2}}{r^2}B_1 = -i\epsilon h_0.$$

Note that all terms with the variable B_1 cancel each other, so we arrive at the equation

$$\frac{a\sqrt{2}}{r}h_1+(\frac{d}{dr}+\frac{2}{r})h_2=-i\epsilon h_0; \qquad (2)$$

only the variables $h_0, h_1=h_3, h_2$ enter into eq. (2). Besides, remain two equations which relate the main variables to the gauge ones

$$2)\quad E_2=\frac{1}{i\epsilon}[h_2+\frac{a\sqrt{2}}{r}B_1],\quad 3)\quad E_1=\frac{1}{i\epsilon}[h_1-B_1'-\frac{B_1}{r}].$$

Now consider equations 4), 5), 6):

$$4)\quad -i\epsilon h_1+\frac{1}{2}\frac{a\sqrt{2}}{r}h_0=0 \Rightarrow h_1=\frac{1}{i\epsilon}\frac{a}{\sqrt{2}}\frac{1}{r}h_0,$$

$$5)\qquad -i\epsilon h_2-h_0'=0\ \Rightarrow\ h_2=-\frac{1}{i\epsilon}h_0',$$

$$6)\quad h_1'+\frac{h_1}{r}+\frac{1}{2}\frac{a\sqrt{2}}{r}h_2=0;$$

let us substitute expressions for h_1 and h_2 into equation 6), which yields the identity

$$7)\quad \frac{d}{dr}\frac{1}{i\epsilon}\frac{a}{\sqrt{2}}\frac{1}{r}h_0+\frac{1}{r}\frac{1}{i\epsilon}\frac{a}{\sqrt{2}}\frac{1}{r}h_0-\frac{a}{\sqrt{2}}\frac{1}{r}\frac{1}{i\epsilon}\frac{d}{dr}h_0=0\quad\Rightarrow\quad 0=0;$$

so equation 6) may be ignored. Thus we have only three independent equations

$$\frac{a\sqrt{2}}{r}h_1+\frac{d}{dr}h_2+\frac{2}{r}h_2=-i\epsilon h_0,\quad h_1=\frac{1}{i\epsilon}\frac{a}{\sqrt{2}}\frac{1}{r}h_0,\quad h_2=-\frac{1}{i\epsilon}h_0'. \qquad (3)$$

Eliminating the variables h_1 and h_2, we get

$$\frac{a\sqrt{2}}{r}\frac{1}{i\epsilon}\frac{a}{\sqrt{2}}\frac{1}{r}h_0-\frac{d}{dr}\frac{1}{i\epsilon}\frac{d}{dr}h_0-\frac{2}{r}\frac{1}{i\epsilon}\frac{d}{dr}h_0=-i\epsilon h_0,$$

whence we obtain the second order equation for h_0:

$$(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}+\epsilon^2-\frac{a^2}{r^2})h_0=0.$$

Therefore, instead of (3) we get the equivalent system

$$(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}+\epsilon^2-\frac{a^2}{r^2})h_0=0,\quad h_1=\frac{1}{i\epsilon}\frac{a}{\sqrt{2}}\frac{1}{r}h_0,\quad h_2=-\frac{1}{i\epsilon}h_0'; \qquad (4)$$

recall two equations

$$2)\quad E_2=\frac{1}{i\epsilon}[h_2-h'+\frac{a\sqrt{2}}{r}B_1],\quad 3)\quad E_1=\frac{1}{i\epsilon}[h_1+\frac{1}{2}\frac{a\sqrt{2}}{r}h-B_1'-\frac{B_1}{r}].$$

Two last equations may be rewritten differently

$$i\epsilon E_2-\frac{a\sqrt{2}}{r}B_1=h_2-h',\quad i\epsilon E_1+B_1'+\frac{B_1}{r}=h_1+\frac{1}{2}\frac{a\sqrt{2}}{r}h.$$

In accordance with eq. (4), we may express h_1 and h_2 through the variable h_0, then we obtain

$$i\epsilon E_2 - \frac{a\sqrt{2}}{r} B_1 = -\frac{1}{i\epsilon} h_0' - h' = -(\frac{1}{i\epsilon} h_0' + h'),$$

$$i\epsilon E_1 + B_1' + \frac{B_1}{r} = \frac{1}{i\epsilon}\frac{a}{\sqrt{2}}\frac{1}{r} h_0 + \frac{1}{2}\frac{a\sqrt{2}}{r} h = \frac{a}{\sqrt{2}r}(\frac{1}{i\epsilon} h_0 + h).$$

Therefore, the problem is reduced to 3 equations:

$$(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + \epsilon^2 - \frac{a^2}{r^2}) h_0 = 0,$$

$$i\epsilon E_2 - \frac{a\sqrt{2}}{r} B_1 = -\frac{d}{dr}\frac{1}{i\epsilon} h_0,$$

$$i\epsilon E_1 + B_1' + \frac{B_1}{r} = \frac{a}{\sqrt{2}r}\frac{1}{i\epsilon} h_0.$$

For the last system, we can see 2 independent solutions:

$$I, \quad \begin{array}{l} h_0 \neq 0, \qquad (\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + \epsilon^2 - \frac{a^2}{r^2}) h_0 = 0, \\ i\epsilon E_2 - \frac{a\sqrt{2}}{r} B_1 = -\frac{1}{i\epsilon}\frac{d}{dr} h_0, \quad i\epsilon E_1 + B_1' + \frac{B_1}{r} = \frac{a}{\sqrt{2}r}\frac{1}{i\epsilon} h_0; \end{array}$$

$$II, \qquad h_0 = 0, \quad i\epsilon E_2 - \frac{a\sqrt{2}}{r} B_1 = 0, \quad i\epsilon E_1 + B_1' + \frac{B_1}{r} = 0.$$

Thus, the system of 6 equations with the parity $P = (-1)^j$ has 2 independent solutions; the first one I includes both the gauge and the physical variables; the second one II is purely gauge. Recall that in the case of the parity $P = (-1)^{J+1}$ we had found 2 pure gauge solutions (see (1)).

Recall the initial substitution for field function [5]

$$\bar{H}_1 = e^{-i\epsilon t} \begin{vmatrix} h_0(r) D_0 \\ h_1(r) D_{-1} \\ h_2(r) D_0 \\ h_3(r) D_{+1} \end{vmatrix}, \quad \bar{H}_2 = e^{-i\epsilon t} \begin{vmatrix} E_1(r) D_{-1} \\ E_2(r) D_0 \\ E_3(r) D_{+1} \\ B_1(r) D_{+1} \\ B_2(r) D_0 \\ B_3(r) D_{-1} \end{vmatrix}. \qquad (5)$$

From (5), for solutions with the parity $P = (-1)^{j+1}$, we get

$$h_0=0, h_2=0, h_3=-h_1, E_3=-E_1, E_2=0, B_3=+B_1, \quad \bar{H}_1=e^{-i\epsilon t}\begin{vmatrix} 0 \\ h_1 D_{-1} \\ 0 \\ h_1 D_{+1} \end{vmatrix}, \bar{H}_2=e^{-i\epsilon t}\begin{vmatrix} E_1 D_{-1} \\ 0 \\ -E_1 D_{+1} \\ B_1 D_{+1} \\ B_2 D_0 \\ B_1 D_{-1} \end{vmatrix};$$

here we can see two independent purely gauge solutions

$$1. \quad h_1=0, \quad any\ B_1, B_2=0, \quad E_1, \quad i\epsilon E_1=(\frac{d}{dr}+\frac{1}{r})B_1;$$

$$2. \quad h_1=0, \quad B_1=0,\ any\ B_2, \quad E_1, \quad i\epsilon E_1=-\frac{a}{\sqrt{2}}B_2;$$

each of them is determined by an arbitrary function: by B_1 and B_2.

From (5), for states with the parity $P=(-1)^j$ we obtain

$$h_3=+h_1, B_3=-B_1, B_2=0, E_3=+E_1, \qquad \bar{H}_1=e^{-i\epsilon t}\begin{vmatrix} h_0(r)D_0 \\ h_1 D_{-1} \\ h_2(r)D_0 \\ h_1 D_{+1} \end{vmatrix}, \quad \bar{H}_2=e^{-i\epsilon t}\begin{vmatrix} E_1 D_{-1} \\ E_2 D_0 \\ E_1 D_{+1} \\ B_1 D_{+1} \\ 0 \\ -B_1 D_{-1} \end{vmatrix};$$

here we can see two independent solutions, which are determined by relations below

$$\text{I,} \qquad \begin{aligned} &(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}+\epsilon^2-\frac{a^2}{r^2})h_0=0, \\ &i\epsilon\bar{E}_2-\frac{a\sqrt{2}}{r}\bar{B}_1=-\frac{1}{i\epsilon}\frac{d}{dr}h_0, \quad i\epsilon\bar{E}_1+\bar{B}'_1+\frac{\bar{B}_1}{r}=\frac{a}{\sqrt{2}r}\frac{1}{i\epsilon}h_0; \end{aligned}$$

$$\text{II,} \qquad h_0=0, \qquad i\epsilon E_2-\frac{a\sqrt{2}}{r}B_1=0, \quad i\epsilon E_1+B_1'+\frac{B_1}{r}=0.$$

the first one I includes both the gauge and physical variables; the second one II is purely gauge.

Conclusions

The system of equations for Calb – Ramond particle has been studied in spherical coordinates. After separating the variables we have derived the system of 10 radial equations. We have constructed exact solutions of this system. It is shown that for the Calb – Ramond particle there exist only one physically observable state with spherical symmetry, and three states are gauge ones. This situation correlates with what we have for solutions with Cartesian and cylindrical symmetries. Recall that in Maxwell theory, there exist two physically observable states, and two pure gauge states, so the Calb – Ramond field substantially differs from the Maxwell theory.

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