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Alina Ivashkevich[1], Anton Buryy[2], Elena Ovsiyuk[3], Victor Red'kov[4]
[1, 2]Graduate Student of B. I. Stepanov Institute of Physics
of the National Academy of Sciences of Belarus
[3]PhD (Physics and Mathematics), Assistant Professor,
Head of the Department of Theoretical Physics and Applied Informatics
of the Mozyr State Pedagogical University named after I. P. Shamyakin
[4]PhD (Physics and Mathematics), Chief Researcher
of the Center «Fundamental Interactions and Astrophysics»
of the B. I. Stepanov Institute of Physics
of the National Academy of Sciences of Belarus
Алина Валентиновна Ивашкевич[1], Антон Васильевич Бурый[2],
Елена Михайловна Овсиюк[3], Виктор Михайлович Редьков[4]
[1, 2]аспирант Института физики имени Б. И. Степанова НАН Беларуси
[3]канд. физ.-мат. наук, зав. каф. теоретической физики и прикладной информатики
Мозырского государственного педагогического университета имени И. П. Шамякина
[4]д-р физ.-мат. наук, гл. науч. сотрудник
Института физики имени Б. И. Степанова НАН Беларуси
e-mail: [1]ivashkevich.alina@yandex.by; [2]anton.buryy.97@mail.ru; [3]e.ovsiyuk@mail.ru;
[4]v.redkov@ifanbel.bas-net.by

Eigenstates of the Helicity Operator for a Particle with Spin 2 in an External Magnetic Field

The helicity operator for a symmetric tensor of the second rank describing a particle with spin 2 is detailed in a cylindrical coordinate system and a cylindrical tetrad. First, the case of a free particle is investigated. After separating the variables, a system of 10 differential equations of the first order for the variable r is obtained. It is divided into two subsystems of 4 and 6 equations. A system of 4 equations is solved using the exclusion method and reduced to a single equation for the main function; its solutions are constructed in terms of Bessel functions. A system of 6 equations is reduced to two related second-order equations for two functions, which give a 4th-order differential equation; the latter is solved by factorization; exact solutions are constructed in Bessel functions. The whole analysis is generalized to the case of the presence of an external magnetic field. Solutions are constructed in degenerate hypergeometric functions. The spectra for the eigenstates of the helicity operator are described as the roots of polynomial equations of the 3rd and 5th orders.

Key words: *spin 2; cylindrical symmetry, tetrad method, helicity operator, eigenvalue problem, exact solutions.*

СОБСТВЕННЫЕ СОСТОЯНИЯ ОПЕРАТОРА СПИРАЛЬНОСТИ ДЛЯ ЧАСТИЦЫ СО СПИНОМ 2 ВО ВНЕШНЕМ МАГНИТНОМ ПОЛЕ

Оператор спиральности для симметричного тензора второго ранга, описывающего частицу со спином 2, детализируется в цилиндрической системе координат и цилиндрической тетраде. Сначала исследуется случай свободной частицы. После разделения переменных получена система из 10-ти дифференциальных уравнений первого порядка по переменной r. Она разбивается на две подсистемы из 4-х и 6-ти уравнений. Система из 4-х уравнений решается с помощью метода исключения и сводится к одному уравнению для основной функции, его решения строятся в терминах функций Бесселя. Система из 6-ти уравнений приводится к двум связанным уравнениям второго порядка для двух функций, которые дают дифференциальное уравнение 4-го порядка; последнее решается методом факторизации; точные решения строятся в функциях Бесселя. Весь анализ обобщается на случай присутствия внешнего магнитного поля. Решения строятся в вырожденных гипергеометрических функциях. Спектры для собственных состояний оператора спиральности описываются как корни полиномиальных уравнений 3-й и 5-й степеней.

Ключевые слова: *спин 2,симметричный тензор 2-го ранга, цилиндрическая симметрия, тетрадный метод, оператор спиральности, задача на собственные значения, точные решения.*

Introduction, helicity operator in cylindric basis, separating the variables

It is known that the eigenvalues states of the helicity operator play a substantial role in studying any spin particle in external electromagnetic (or gravitational) fields with cylindric symmetry. In the present work we specify this problem for a spin 2 particle in Minkowski space-time, starting with the following relations in Cartesian coordinates

$$\Sigma^{cart} = J^{23}\frac{\partial}{\partial x} + J^{31}\frac{\partial}{\partial y} + J^{12}\frac{\partial}{\partial z}, \quad \Sigma^{Cart} H^{cart} = \sigma H^{cart}, \tag{1}$$

$H^{Cart}(x,y,z)$ consists of 10 components of the symmetric tensor referring to the spin 2 particle.

We will apply description of this field in cylindric coordinates $x^{\alpha} = (t, r, \phi, z)$ and the corresponding tetrad

$$dS^2 = dt^2 - dr^2 - r^2 d\phi^2 - dz^2. \tag{2}$$

Transition from Cartesian tetrad to cylindric one is performed by means of the local transformation (belonging to the Lorentz group)

$$L_b^{\ a}(x) = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & \sin\phi & 0 \\ 0 & -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}. \tag{3}$$

Therefore, the tensor of the second rank H transforms according to the rule

$$H^{cart} = [H^{cart}_{(ab)}], \quad H^{cyl} = (L \otimes L) H^{cart} = L H^{cart} \tilde{L}.$$

Correspondingly the helicity operator transforms to cylindric tetrad as follows (we apply 10-dimensional representation for S)

$$H^{cyl} = (L \otimes L) H^{cart} \Rightarrow H^{cyl} = S H^{cart},$$

$$\Sigma^{cyl} = S(\phi)[J^{23}(\cos\phi\frac{\partial}{\partial r} - \frac{\sin\phi}{r}\frac{\partial}{\partial \phi}) + J^{31}(\sin\phi\frac{\partial}{\partial r} + \frac{\cos\phi}{r}\frac{\partial}{\partial \phi}) + J^{12}\frac{\partial}{\partial z}]S(-\phi), \tag{4}$$

whence we get

$$\Sigma^{cyl} = [S(\phi)J^{23}S_2(-\phi)\cos\phi + S_2(\phi)J^{31}S(-\phi)\sin\phi]\frac{\partial}{\partial r}$$
$$+\frac{1}{r}[-S(\phi)J^{23}S(-\phi)\sin\phi + S_2(\phi)J^{31}S_2(-\phi)\cos\phi]\frac{\partial}{\partial \phi}$$
$$+\frac{1}{r}[-\sin\phi S(\phi)J^{23}\frac{\partial S(-\phi)}{\partial \phi} + \cos\phi S(\phi)J^{31}\frac{\partial S(-\phi)}{\partial \phi}] + S(\phi)J^{12}S(-\phi)\frac{\partial}{\partial z}.$$

Taking in mind the structure of the field function (it contains the multipliers $e^{im\phi}$ and e^{ikz}) we can transform the last relation to the form

$$\Sigma^{cyl} = [S(\phi)J^{23}S(-\phi)\cos\phi + S(\phi)J^{31}S(-\phi)\sin\phi]\frac{d}{dr} + \frac{im}{r}[-S(\phi)J^{23}S(-\phi)\sin\phi + S(\phi)J^{31}S(-\phi)\cos\phi]$$
$$+\frac{1}{r}[-\sin\phi S(\phi)J^{23}\frac{\partial S(-\phi)}{\partial \phi} + \cos\phi S(\phi)J^{31}\frac{\partial S(-\phi)}{\partial \phi}] + S(\phi)J^{12}S(-\phi)ik.$$

Let us find expression for 10 components of the field function H^{cyl} in the cylindric tetrad basis. We will apply the following notations

$$H^{cyl}=\begin{vmatrix} f_1\\ f_2\\ f_3\\ c_1\\ c_2\\ c_3\\ d_1\\ d_2\\ d_3\\ f_0 \end{vmatrix},\ S(\phi)=\begin{vmatrix} \cos^2\phi & \sin^2\phi & . & . & . & \sin 2\phi & . & . & . & .\\ \sin^2\phi & \cos^2\phi & . & . & . & -\sin 2\phi & . & . & . & .\\ . & . & 1 & . & . & . & . & . & . & .\\ . & . & . & \cos\phi & -\sin\phi & . & . & . & . & .\\ . & . & . & \sin\phi & \cos\phi & . & . & . & . & .\\ -\cos\phi\sin\phi & \cos\phi\sin\phi & . & . & . & \cos 2\phi & . & . & . & .\\ . & . & . & . & . & . & \cos\phi & \sin\phi & . & .\\ . & . & . & . & . & . & -\sin\phi & \cos\phi & . & .\\ . & . & . & . & . & . & . & . & 1 & .\\ . & . & . & . & . & . & . & . & . & 1 \end{vmatrix};$$

the inverse matrix equals $S^{-1}=S(-\phi)$. After simple calculation we derive the following expression for helicity operator in cylindric basis

$$\Sigma^{cyl}=\begin{vmatrix} 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&-2&0&0&0&0&0&0\\ 0&0&0&2&0&0&0&0&0&0\\ 0&1&-1&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&-1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&-1&0\\ 0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&0 \end{vmatrix}\frac{d}{dr}+\frac{im}{r}\begin{vmatrix} 0&0&0&0&2&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&-2&0&0&0&0&0\\ 0&0&0&0&0&-1&0&0&0&0\\ -1&0&1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&-1&0&0&0\\ 0&0&0&0&0&0&0&0&0&0 \end{vmatrix}$$

$$+ik\begin{vmatrix} 0&0&0&0&0&-2&0&0&0&0\\ 0&0&0&0&0&2&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0\\ 0&0&0&-1&0&0&0&0&0&0\\ 1&-1&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&-1&0&0\\ 0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0 \end{vmatrix}+\begin{vmatrix} 0&0&0&-2&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&2&0&0&0&0&0&0\\ -1&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&2&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&0 \end{vmatrix},$$

taking in mind explicit form of generators for tensor we obtain its shorter form

$$\Sigma^{cyl} = J^{23}\frac{d}{dr} + \frac{im}{r}J^{31} + ikJ^{12} + \frac{3}{1}r\begin{vmatrix} 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix}. \qquad (5)$$

We will use the cyclic representation, in which the generator J^{12} becomes diagonal, this leads to

$$\bar\Sigma^{cyl} = \bar J^{23}\frac{d}{dr} + \frac{im}{r}\bar J^{31} + ik\bar J^{12} + \frac{1}{r}\begin{vmatrix} 0 & 0 & 0 & 0 & 0 & i\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i\sqrt{2} & 0 & -i\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -i\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ -i\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix},$$

where (see [29])

$$\bar J^{23} = \frac{1}{\sqrt{2}}\begin{vmatrix} . & . & . & . & . & -2i & . & . & . & . \\ . & . & . & -2i & . & -2i & . & . & . & . \\ . & . & . & -2i & . & . & . & . & . & . \\ . & -i & -i & . & -i & . & . & . & . & . \\ . & . & . & -i & . & -i & . & . & . & . \\ -i & -i & . & . & -i & . & . & . & . & . \\ . & . & . & . & . & . & . & -i & . & . \\ . & . & . & . & . & . & -i & . & -i & . \\ . & . & . & . & . & . & . & -i & . & . \\ . & . & . & . & . & . & . & . & . & . \end{vmatrix}, \qquad (6)$$

$$\bar{J}^{31}=\frac{1}{\sqrt{2}}\begin{vmatrix} . & . & . & . & . & -2 & . & . & . & . \\ . & . & . & -2 & . & 2 & . & . & . & . \\ . & . & . & 2 & . & . & . & . & . & . \\ . & 1 & -1 & . & 1 & . & . & . & . & . \\ . & . & . & -1 & . & 1 & . & . & . & . \\ 1 & -1 & . & . & -1 & . & . & . & . & . \\ . & . & . & . & . & . & . & -1 & . & . \\ . & . & . & . & . & . & 1 & . & -1 & . \\ . & . & . & . & . & . & . & 1 & . & . \\ . & . & . & . & . & . & . & . & . & . \end{vmatrix},\quad \bar{J}^{12}=\begin{vmatrix} -2i & . & . & . & . & . & . & . & . & . \\ . & 0 & . & . & . & . & . & . & . & . \\ . & . & +2i & . & . & . & . & . & . & . \\ . & . & . & +i & . & . & . & . & . & . \\ . & . & . & . & 0 & . & . & . & . & . \\ . & . & . & . & . & -i & . & . & . & . \\ . & . & . & . & . & . & -i & . & . & . \\ . & . & . & . & . & . & . & 0 & . & . \\ . & . & . & . & . & . & . & . & +i & . \\ . & . & . & . & . & . & . & . & . & 0 \end{vmatrix}, \qquad (7)$$

The eigenvalue equation $\bar{\Sigma}\bar{H}=\sigma\bar{H}$ gives the following differential system of 11 equations

$$2c_{3'}=-\frac{2(m-1)}{r}c_3+i\sqrt{2}(\sigma-2k)f_1,\ 2\left(c_{1'}+c_{3'}\right)=-\frac{2(m+1)}{r}c_1+\frac{2(m-1)}{r}c_3+i\sqrt{2}\sigma f_2,$$

$$2c_{1'}=i\sqrt{2}(2k+\sigma)f_3+\frac{2(m+1)}{r}c_1,\ c_{2'}+f_{2'}+f_{3'}=i\sqrt{2}(k+\sigma)c_1+\frac{m}{r}\left(c_2+f_2\right)-\frac{m+2}{r}f_3,$$

$$c_{1'}+c_{3'}=-\frac{m+1}{r}c_1+i\sqrt{2}\sigma c_2+\frac{m-1}{r}c_3,\ c_{2'}+f_{1'}+f_{2'}=i\sqrt{2}(\sigma-k)c_3+\frac{m-2}{r}f_1-\frac{m}{r}\left(c_2+f_2\right),$$

$$d_{2'}=i\sqrt{2}(\sigma-k)d_1-\frac{m}{r}d_2,\ d_{1'}+d_{3'}=\frac{m-1}{r}d_1+i\sqrt{2}\sigma d_2-\frac{m+1}{r}d_3,\ d_{2'}=i\sqrt{2}(k+\sigma)d_3+\frac{m}{r}d_2,$$

the 10-th equation is trivial, $\sqrt{2}\sigma f_0=0$. It is convenient to apply special notations for 8 operators:

$$\frac{1}{\sqrt{2}}(\frac{d}{dr}\pm\frac{m}{r})=a_m^{\pm},\quad \frac{1}{\sqrt{2}}(\frac{d}{dr}\pm\frac{m+1}{r})=a_{m+1}^{\pm},\quad \frac{1}{\sqrt{2}}(\frac{d}{dr}\pm\frac{m-1}{r})=a_{m-1}^{\pm},$$
$$\frac{1}{\sqrt{2}}(\frac{d}{dr}+\frac{m+2}{r})=a_{m+2}^{+},\qquad \frac{1}{\sqrt{2}}(\frac{d}{dr}-\frac{m-2}{r})=a_{m-2}^{-}. \qquad (8)$$

Then the above system is presented as three independent subsystems:

$$I\qquad a_m^{+}d_2=i(\sigma-k)d_1,\quad a_{m-1}^{-}d_1+a_{m+1}^{+}d_3=i\sigma d_2,\quad a_m^{-}d_2=i(\sigma+k)d_3; \qquad (9)$$

$$II\qquad a_{m-1}^{+}c_3=i(\frac{\sigma}{2}-k)f_1,\quad a_{m+1}^{+}c_1+a_{m-1}^{-}c_3=i\frac{\sigma}{2}f_2,$$

$$a_{m+1}^{-}c_1=i(\frac{\sigma}{2}+k)f_3,\quad a_m^{-}c_2+a_m^{-}f_2+a_{m+2}^{+}f_3=i(\sigma+k)c_1, \qquad (10)$$

$$a_{m+1}^{+}c_1+a_{m-1}^{-}c_3=i\sigma c_2,\quad a_m^{+}c_2+a_{m-2}^{-}f_1+a_m^{+}f_2=i(\sigma-k)c_3;$$

$$III\qquad \sigma f_0=0\ \Rightarrow\ \sigma\neq 0,\ f_0=0. \qquad (11)$$

Consider the system I; after eliminating the variables d_1 and d_3 we get one 2-nd order equation for the primary function

$$\left[\frac{1}{i(\sigma-k)}a_{m-1}^{-}a_m^{+}+\frac{1}{i(\sigma+k)}a_{m+1}^{+}a_m^{-}-i\sigma\right]d_2=0. \qquad (12)$$

Similarly, in the system II one can eliminate the variables f_1, f_2, f_3:

$$f_1 = \frac{2}{i(\sigma - 2k)} a^+_{m-1} c_3, \quad f_2 = \frac{2}{i\sigma} a^+_{m+1} c_1 + \frac{2}{i\sigma} a^-_{m-1} c_3, \quad f_3 = \frac{2}{i(\sigma + 2k)} a^-_{m+1} c_1,$$

this results in the system for c_1, c_2, c_3:

$$a^-_m c_2 + \frac{2}{i\sigma} a^-_m a^+_{m+1} c_1 + \frac{2}{i\sigma} a^-_m a^-_{m-1} c_3 + \frac{2}{i(\sigma+2k)} a^+_{m+2} a^-_{m+1} c_1 = i(\sigma + k) c_1,$$

$$a^+_m c_2 + \frac{2}{i(\sigma-2k)} a^-_{m-2} a^+_{m-1} c_3 + \frac{2}{i\sigma} a^+_m a^+_{m+1} c_1 + \frac{2}{i\sigma} a^+_m a^-_{m-1} c_3 = i(\sigma - k) c_3,$$

$$a^+_{m+1} c_1 + a^-_{m-1} c_3 = i\sigma c_2.$$

With the use of third equation we may eliminate the variable c_2, so deriving the system for c_1, c_3:

$$\begin{aligned} &[3a^-_m a^+_{m+1} + \frac{2\sigma}{(\sigma+2k)} a^+_{m+2} a^-_{m+1} + \sigma(\sigma+k)] c_1 + 3a^-_m a^-_{m-1} c_3 = 0, \\ &[3a^+_m a^-_{m-1} c_3 + \frac{2\sigma}{(\sigma-2k)} a^-_{m-2} a^+_{m-1} + \sigma(\sigma-k)] c_3 + 3a^+_m a^+_{m+1} c_1 = 0. \end{aligned} \tag{13}$$

In order to take into account the presence of external magnetic field, it suffices to make one change, $m \Rightarrow m + eBr^2/2$, this leads to new 8 operators instead of operators (8).

The system I, the free particle

Let us turn to the case I, first consider eq. (12) for the free particle. Allowing for the identities

$$a^-_{m-1} a^+_m = \frac{1}{2}\frac{d^2}{dr^2} + \frac{1}{2r}\frac{d}{dr} - \frac{m^2}{2r^2}, \qquad a^+_{m-1} a^-_m = \frac{1}{2}\frac{d^2}{dr^2} - \frac{1}{2r}\frac{d}{dr} + \frac{(2-m)m}{2r^2}$$

we get

$$\left[\frac{d^2}{dr^2} + \frac{k}{\sigma r}\frac{d}{dr} - \frac{mk + m(m-1)\sigma}{\sigma r^2} - \left(k^2 - \sigma^2\right)\right] d_2 = 0. \tag{14}$$

Near the point $r = 0$, this equation becomes simpler

$$[\frac{d^2}{dr^2} + \frac{k}{\sigma r}\frac{d}{dr} - \frac{mk + m(m-1)\sigma}{\sigma r^2}] d_2 = 0, \quad d_2 = r^A, \quad A = m, -\frac{k}{\sigma} - m + 1.$$

At infinity, $r \to \infty$, we have

$$[\frac{d^2}{dr^2} + \frac{k}{\sigma r}\frac{d}{dr} - (k^2 - \sigma^2)] d_2 = 0, \quad d_2 = e^{\pm i\sqrt{\sigma^2 - k^2} r}. \tag{15}$$

Therefore, solutions should be searched in the form $d_2 = r^A e^{Cr} f$. Imposing the known restrictions on A and C, we get

$$A = m, -\frac{k}{\sigma} - m + 1; C = \pm i\sqrt{\sigma^2 - k^2}, r\frac{d^2 f}{dr^2} + (2A + \frac{k}{\sigma} + 2Cr)\frac{df}{dr} + (2AC + \frac{k}{\sigma}C) f = 0. \tag{16}$$

Changing the variable, $z = -2Cr$, we reduce the equation to hypergeometric form (let $C = +i\sqrt{\sigma^2 - k^2}$):

$$z\frac{d^2 f}{dz^2}+(2A+\frac{k}{\sigma}-z)\frac{df}{dz}-(A+\frac{k}{2\sigma})f=0,$$
$$d_2=z^A e^{-z/2}\Phi(c,a,z),\quad a=A+\frac{k}{2\sigma},\quad c=2A+\frac{k}{\sigma}=2a. \tag{17}$$

Solutions will be regular in the point $r=0$, if we take positive A; we should distinguish two cases:

$$(a)\qquad m>0,\quad d_2\sim z^A=z^m\to 0;$$
$$(b)\qquad m<0,\quad d_2\sim z^A=z^{1-m-\frac{k}{\sigma}},\quad 1-m>k/\sigma. \tag{18}$$

Solutions under consideration may be presented in terms of Bessel functions. Indeed, starting with the equation (14), let us make the substitution $d_2(r)=\varphi(r)\bar{d}_2(r)$, then we obtain

$$\bar{d}''_2+(2\frac{\varphi'}{\varphi}+\frac{k}{\sigma r})\bar{d}'_2+(\frac{k}{\sigma r}\frac{\varphi'}{\varphi}+\frac{\varphi''}{\varphi}-\frac{mk+m(m-1)\sigma}{\sigma r^2}-(k^2-\sigma^2))\bar{d}_2=0.$$

The function φ is fixed as follows

$$2\frac{\varphi'}{\varphi}+\frac{k}{\sigma r}=\frac{1}{r}\ \Rightarrow\ \varphi=r^{(1-k/\sigma)/2}; \tag{19}$$

accordingly the above equation takes the form

$$\bar{d}''_2+\frac{1}{r}\bar{d}'_2+(-(k^2-\sigma^2)-\frac{(m-1/2+k/2\sigma)^2}{r^2})\bar{d}_2=0.$$

In the variable $y=i\sqrt{k^2-\sigma^2}\,r$, it has the structure of Bessel equation

$$(\frac{d^2}{dy^2}+\frac{1}{y}\frac{d}{dy}+1-\frac{\nu^2}{y^2})\bar{d}'_2=0,\quad \nu=m-\frac{1}{2}+\frac{k}{2\sigma},\quad d_2(y)=r^{\frac{1-k/\sigma}{2}}\bar{d}_2(z). \tag{20}$$

The case I, particle in magnetic field

In presence of the magnetic field, the equation for d_2 takes the form (let $eB\Rightarrow B$)

$$[\frac{d^2}{dr^2}+\frac{k}{r\sigma}\frac{d}{dr}+\frac{-2\sigma\left(B(2m-1)+2k^2\right)+2Bk+4\sigma^3}{4\sigma}-\frac{B^2}{4}r^2+\frac{-km-(m-1)m\sigma}{r^2\sigma}]d_2=0. \tag{21}$$

Near the point $r=0$, we get

$$[\frac{d^2}{dr^2}+\frac{k}{r\sigma}\frac{d}{dr}-\frac{km+(m-1)m\sigma}{\sigma r^2\sigma}]d_2=0,$$

which coincides with that in the case of a free particle. At infinity we have

$$(\frac{d^2}{dr^2}+\frac{k}{r\sigma}\frac{d}{dr}-\frac{1}{4}B^2r^2)d_2=0,\quad d_2=e^{Cr^2},\quad C=\pm\frac{B}{4}.$$

In the new variable, $r^2=x$, the above equation reads

$$[\frac{d^2}{dx^2}+\frac{(k+\sigma)}{2\sigma x}\frac{d}{dx}-\frac{1}{16}B^2+\frac{B(k-2m\sigma+\sigma)}{8\sigma x}-\frac{(k^2-\sigma^2)}{4x}+\frac{m(\sigma-k)}{4\sigma x^2}-\frac{m^2}{4x^2}]d_2=0. \tag{22}$$

Its solutions are searched in the form $d_2 = x^A e^{Cx} f(x)$, imposing the known constraints

$$A = \frac{m}{2}, \frac{1-m}{2} - \frac{k}{2\sigma}, \quad and \quad C = -\frac{B}{4}, +\frac{B}{4} \tag{23}$$

(below assume $C = -\frac{B}{4}, B > 0)$, we reduce equation for $f(x)$ to the form

$$xf'' + (2A + \frac{(k+\sigma)}{2\sigma} + 2Cx) f' + [\frac{B(k - 2m\sigma + \sigma) + 4C(k+\sigma)}{8\sigma} - \frac{(k^2 - \sigma^2) - 8AC}{4}] f = 0.$$

In the variable $2Cx = -z$, it reads as an equation of confluent hypergeometric type

$$zf'' + (c - z) f' - af = 0, \quad z\frac{d^2 f''}{dz^2} + (2A + \frac{k+\sigma}{2\sigma} - z)\frac{df}{dz} - (\frac{m}{2} + \frac{k^2 - \sigma^2}{2B} + A) f = 0. \tag{24}$$

Imposing the usual constraint to get polynomials

$$a = \frac{m}{2} + \frac{k^2 - \sigma^2}{2B} + A = -n, \quad n = 0,1,2,3,...,$$

we find the quantization rule for σ:

$$\sigma = \pm\sqrt{k^2 + (A + \frac{m}{2} + n)2B}. \tag{25}$$

Depending on the sign of m, we distinguish two cases:

$$m > 0, \quad \sigma = \pm\sqrt{k^2 + (m+n)2B}, \qquad m < 0, \quad \sigma = \pm\sqrt{k^2 + (n + \frac{1 - k/\sigma}{2})2B}, \tag{26}$$

the second equation determines σ in inexplicit form. Assuming that solutions are regular at the point $z = 0$, for function d_2 we have two different representations depending on the sign of m:

$$m > 0, \quad d_2 \sim z^{m/2} e^{-z/2} \Phi(-n, c, z), \quad c = m + \frac{1}{2} + \frac{k}{2\sigma}; \tag{27}$$

$$m < 0, \quad d_2 \sim z^{\frac{1-m}{2} - \frac{k}{2\sigma}} e^{-z/2} \Phi(-n, c', z), \quad c = \frac{3}{2} - m, \frac{1-m}{2} > \frac{k}{2\sigma}. \tag{28}$$

Note that the second equation for σ in (26) leads to a cubic equation

$$\sigma^3 - \sigma[(k^2 + B(2n+1)] + Bk = 0, \quad B > 0; \tag{29}$$

in the dimensionless variables $\sigma / k = X, B / k^2 = b$ it reads simpler, $X^3 - X[1 + b(2n+1)] + b = 0$.

The case II, free particle

Let us turn to the case II for the free particle. The system (13) in explicit form reads

$$[(3 + \frac{2\sigma}{\sigma + 2k})(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{(m+1)^2}{r^2}) + 2\sigma(\sigma + k)]c_1 + 3(\frac{d^2}{dr^2} + \frac{(1-2m)}{r}\frac{d}{dr} + \frac{(m^2-1)}{r^2})c_3 = 0,$$

$$[(3 + \frac{2\sigma}{\sigma - 2k})(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{(m-1)^2}{r^2}) + 2\sigma(\sigma - k)]c_3 + 3(\frac{d^2}{dr^2} + \frac{(1+2m)}{r}\frac{d}{dr} + \frac{(m^2-1)}{r^2})c_1 = 0.$$

This system may be considered as linear with respect to the variables c_1'', c_3'':

$$\begin{vmatrix} \begin{vmatrix} 3+\frac{2\sigma}{2k+\sigma} & 3 \\ 3 & 3-\frac{2\sigma}{2k-\sigma} \end{vmatrix} \begin{vmatrix} c_1'' \\ c_3'' \end{vmatrix} \end{vmatrix}$$

$$= -\begin{vmatrix} [(3+\frac{2\sigma}{\sigma+2k})(\frac{1}{r}\frac{d}{dr}-\frac{(m+1)^2}{r^2})+2\sigma(k+\sigma)]c_1 + 3(\frac{(1-2m)}{r}\frac{d}{dr}+\frac{(m^2-1)}{r^2})c_3 \\ 3(\frac{(1+2m)}{r}\frac{d}{dr}+\frac{(m^2-1)}{r^2})c_1 + [(3+\frac{2\sigma}{\sigma-2k})(\frac{1}{r}\frac{d}{dr}-\frac{(-m+1)^2}{r^2})-2\sigma(k-\sigma)]c_3 \end{vmatrix},$$

evidently it is symmetric under the change $c_3 \Leftrightarrow c_1, m \Leftrightarrow -m, \sigma \Leftrightarrow -\sigma.$ Its solution has the form

$$c_1'' = \frac{1}{8}\frac{(9m-8)\sigma^2 - 36k^2 m}{\sigma^2 r}c_1'$$
$$+[\frac{1}{8}\frac{8\sigma k^2 + 12k^3 - 9\sigma^2 k - 5\sigma^3}{\sigma} - \frac{1}{8}\frac{(m+1)(-17\sigma^2 m - 8\sigma^2 + 36k^2 m)}{\sigma^2 r^2}]c_1$$
$$-\frac{3}{8}\frac{m(-4\sigma k + 12k^2 - 5\sigma^2)}{\sigma^2 r}c_3' + [\frac{3}{8}\frac{(2k^2+\sigma^2-3\sigma k)(\sigma+2k)}{\sigma}$$
$$+\frac{3}{8}\frac{m(-4\sigma mk + 12k^2 m + 5\sigma^2 + 4\sigma k - 12k^2 - 5\sigma^2 m)}{\sigma^2 r^2}]c_3,$$

$$c_3'' = \frac{3}{8}\frac{m(4\sigma k + 12k^2 - 5\sigma^2)}{\sigma^2 r}c_1'$$
$$+[-\frac{3}{8}\frac{(\sigma+k)(-\sigma^2+4k^2)}{\sigma} + \frac{3}{8}\frac{m(6k+5\sigma)(-\sigma+2k)(m+1)}{\sigma^2 r^2}]c_1$$
$$+\frac{1}{8}\frac{36k^2 m - 9\sigma^2 m - 8\sigma^2}{\sigma^2 r}c_3'$$
$$+[-\frac{1}{8}\frac{-9\sigma^2 k + 5\sigma^3 - 8\sigma k^2 + 12k^3}{\sigma} - \frac{1}{8}\frac{(-17\sigma^2 m + 8\sigma^2 + 36k^2 m)(m-1)}{\sigma^2 r^2}]c_3.$$

Shortly we can present the above equations as follows

$$\begin{aligned} \frac{d^2}{dr^2}c_1 &= K_1\frac{d}{dr}c_1 + (\frac{L_1}{r^2}+M_1)c_1 + (\frac{F_1}{r}\frac{d}{dr}+\frac{G_1}{r^2}+H_1)c_3, \\ \frac{d^2}{dr^2}c_3 &= K_3\frac{d}{dr}c_3 + (\frac{L_3}{r^2}+M_3)c_3 + (\frac{F_3}{r}\frac{d}{dr}+\frac{G_3}{r^2}+H_3)c_1. \end{aligned} \tag{30}$$

Let us eliminate from this system the variable $c_3(r)$. To this end, first we present this function as $c_3(r) = \varphi(r)\bar{c}_3(r),$ and impose the constraints

$$(\frac{F_1}{r}\frac{d}{dr}+\frac{G_1}{r^2}+H_1)\varphi\bar{c}_3 = \varphi\frac{F_1}{r}\frac{d}{dr}\bar{c}_3.$$

Whence it follows

$$\frac{F_1}{r}\frac{\varphi'}{\varphi}\bar{c}_3 + \frac{F_1}{r}\bar{c}_3' + \frac{G_1}{r^2}\bar{c}_3 + H_1\bar{c}_3 = \frac{F_1}{r}\frac{d}{dr}\bar{c}_3, \quad or \quad \frac{F_1}{r}\bar{c}_3' + (\frac{F_1}{r}\frac{\varphi'}{\varphi}+\frac{G_1}{r^2}+H_1)\bar{c}_3 = \frac{F_1}{r}\frac{d}{dr}\bar{c}_3.$$

Further we derive equation for determining the function $\varphi(r)$:

$$\frac{F_1}{r}\frac{\varphi'}{\varphi} = -\frac{G_1}{r^2} - H_1 \quad \Rightarrow \quad \ln\varphi = -\frac{G_1}{F_1}\ln r - \frac{H_1}{2F_1}r^2 \quad \Rightarrow \quad \ln(\varphi r^{G_1/F_1}) = -\frac{H_1}{2F_1}r^2,$$

whence it follows $\varphi = r^{-G_1/F_1} e^{-\frac{H_1}{2F_1}r^2}$. Therefore, the initial system may be re-written differently

$$\frac{d^2}{dr^2}c_1 = K_1\frac{d}{dr}c_1 + (\frac{L_1}{r^2} + M_1)c_1 + \varphi\frac{F_1}{r}\frac{d}{dr}\bar{c}_3, \quad c_3 = \varphi\bar{c}_3,$$
$$\frac{d^2}{dr^2}\varphi\bar{c}_3 = K_3\frac{d}{dr}\varphi\bar{c}_3 + (\frac{L_3}{r^2} + M_3)\varphi\bar{c}_3 + (\frac{F_3}{r}\frac{d}{dr} + \frac{G_3}{r^2} + H_3)c_1. \tag{31}$$

From eq. (31), we express the derivative $\frac{d}{dr}\bar{c}_3$:

$$\frac{d}{dr}\bar{c}_3 = \frac{1}{F_1}\frac{r}{\varphi}(\frac{d^2}{dr^2}c_1 - K_1\frac{d}{dr}c_1 - (\frac{L_1}{r^2} + M_1)c_1),$$

below we will need the second derivative as well

$$\frac{d^2}{dr^2}\bar{c}_3 = \frac{d}{dr}\{\frac{1}{F_1}\frac{r}{\varphi}(\frac{d^2}{dr^2}c_1 - K_1\frac{d}{dr}c_1 - (\frac{L_1}{r^2} + M_1)c_1)\}.$$

Expression for function c_3 itself is found by integrating

$$\bar{c}_3 = \int\frac{1}{F_1}\frac{r}{\varphi}(\frac{d^2}{dr^2}c_1 - K_1\frac{d}{dr}c_1 - (\frac{L_1}{r^2} + M_1)c_1)dr.$$

Now we turn to the second equation from (31)

$$\frac{d^2}{dr^2}\varphi\bar{c}_3 = K_3\frac{d}{dr}\varphi\bar{c}_3 + (\frac{L_3}{r^2} + M_3)\varphi\bar{c}_3 + (\frac{F_3}{r}\frac{d}{dr} + \frac{G_3}{r^2} + H_3)c_1;$$

taking into account two identities

$$\frac{d^2}{dr^2}\varphi\bar{c}_3 = \frac{d}{dr}[\varphi'\bar{c}_3 + \varphi\bar{c}'_3] = \varphi''\bar{c}_3 + 2\varphi'\bar{c}'_3 + \varphi\bar{c}''_3, \quad K_3\frac{d}{dr}\varphi\bar{c}_3 = K_3[\varphi'\bar{c}_3 + \varphi\bar{c}'_3],$$

we transform the above equation to other form

$$\varphi''\bar{c}_3 + 2\varphi'\bar{c}'_3 + \varphi\bar{c}''_3 = K_3\varphi'\bar{c}_3 + K_3\varphi\bar{c}'_3 + (\frac{L_3}{r^2} + M_3)\varphi\bar{c}_3 + \frac{F_3}{r}c'_1 + \frac{G_3}{r^2}c_1 + H_3c_1.$$

Whence after re-grouping the terms we obtain

$$\varphi\bar{c}''_3 + (2\varphi' - K_3\varphi)\bar{c}'_3 + (\varphi'' - K_3\varphi' - (\frac{L_3}{r^2} + M_3)\varphi)\bar{c}_3 - \frac{F_3}{r}c'_1 - \frac{G_3}{r^2}c_1 - H_3c_1 = 0,$$

which is equivalent to

$$[\bar{c}''_3 + (2\frac{\varphi'}{\varphi} - K_3)\bar{c}'_3] + (\frac{\varphi''}{\varphi} - K_3\frac{\varphi'}{\varphi} - (\frac{L_3}{r^2} + M_3))\bar{c}_3 - [\frac{1}{\varphi}\frac{F_3}{r}c'_1 + \frac{1}{\varphi}\frac{G_3}{r^2}c_1 + \frac{1}{\varphi}H_3c_1] = 0.$$

With the help of temporary notation

$$\Delta(r) = \frac{\varphi''}{\varphi} - K_3\frac{\varphi'}{\varphi} - (\frac{L_3}{r^2} + M_3),$$

we re-write the last equation differently

$$\frac{1}{\Delta(r)}[\bar{c}'_3+(2\frac{\varphi'}{\varphi}-K_3)\bar{c}_3]+\bar{c}_3-\frac{1}{\Delta(r)}[\frac{1}{\varphi}\frac{F_3}{r}c'_1+\frac{1}{\varphi}\frac{G_3}{r^2}c_1+\frac{1}{\varphi}H_3c_1]=0.$$

After differentiating this equation we derive

$$\frac{d}{dr}\{\frac{1}{\Delta(r)}[\bar{c}'_3+(2\frac{\varphi'}{\varphi}-K_3)\bar{c}_3]\}+\bar{c}'_3-\frac{d}{dr}\{\frac{1}{\Delta(r)}[\frac{1}{\varphi}\frac{F_3}{r}c'_1+\frac{1}{\varphi}\frac{G_3}{r^2}c_1+\frac{1}{\varphi}H_3c_1]\}=0, \quad (32)$$

this is the fourth order differential equation for function $c_1(r)$. In similar manner, we can derive a 4-th order equation for function $c_3(r)$.

Equations of the 4-th order for functions $c_1(r)$ and $c_3(r)$ in explicit form read

$$\frac{d^4c_1}{dr^4}+\frac{2}{r}\frac{d^3c_1}{dr^3}+(\frac{5}{4}\sigma^2-2k^2+\frac{-3-4m-2m^2}{r^2})\frac{d^2c_1}{dr^2}+(\frac{1}{4}\frac{5\sigma^2-8k^2}{r}+\frac{3+4m+2m^2}{r^3})\frac{dc_1}{dr}$$
$$+[-\frac{5}{4}\sigma^2k^2+\frac{1}{4}\sigma^4+k^4+\frac{1}{4}\frac{\left(-5\sigma^2+8k^2\right)(m+1)^2}{r^2}+\frac{(m+3)(m-1)(m+1)^2}{r^4}]c_1=0,$$
$$\frac{d^4c_3}{dr^4}+\frac{2}{r}\frac{d^3c_3}{dr^3}+(\frac{5}{4}\sigma^2-2k^2+\frac{-3+4m-2m^2}{r^2})\frac{d^2c_3}{dr^2}+(\frac{1}{4}\frac{5\sigma^2-8k^2}{r}+\frac{3-4m+2m^2}{r^3})\frac{dc_3}{dr}$$
$$+[-\frac{5}{4}\sigma^2k^2+\frac{1}{4}\sigma^4+k^4+\frac{1}{4}\frac{\left(-5\sigma^2+8k^2\right)(m-1)^2}{r^2}+\frac{(m-3)(m+1)(m-1)^2}{r^4}]c_3=0.$$

Both equations may be factorized:

$$\{\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}+[-k^2+\frac{1}{4}\sigma^2-\frac{(m+1)^2}{r^2}]\}\{\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}+[-k^2+\sigma^2-\frac{(m+1)^2}{r^2}]\}c_1=0,$$
$$\{\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}+[-k^2+\frac{1}{4}\sigma^2-\frac{(m-1)^2}{r^2}]\}\{\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}+[-k^2+\sigma^2-\frac{(m-1)^2}{r^2}]\}c_3=0,$$

where two multipliers are permutable. It suffices to solve 2-nd order equations for c_1:

$$I,[\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-k^2+\sigma^2-\frac{(m+1)^2}{r^2}]c_1=0;\ II,[\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-k^2+\frac{1}{4}\sigma^2-\frac{(m+1)^2}{r^2}]c_1=0. \quad (33)$$

They reduce to Bessel equations

$$I,\quad x=i\sqrt{k^2-\sigma^2}\,r,\quad [\frac{d^2}{dx^2}+\frac{1}{x}\frac{d}{dx}+1-\frac{(m+1)^2}{x^2}]c_1^I(x)=0;$$
$$II,\quad y=i\sqrt{k^2-\frac{\sigma^2}{4}}\,r,\quad [\frac{d^2}{dy^2}+\frac{1}{y}\frac{d}{dy}+1-\frac{(m+1)^2}{y^2}]c_1^{II}(y)=0. \quad (34)$$

Similarly, for function c_3 we get

$$I',\quad x=i\sqrt{k^2-\sigma^2}\,r,\quad [\frac{d^2}{dx^2}+\frac{1}{x}\frac{d}{dx}+1-\frac{(m-1)^2}{x^2}]c_3^I(x)=0;$$
$$II',\quad y=i\sqrt{k^2-\frac{\sigma^2}{4}}\,r,\quad [\frac{d^2}{dy^2}+\frac{1}{y}\frac{d}{dy}+1-\frac{(m-1)^2}{y^2}]c_3^{II}(y)=0. \quad (35)$$

The case II, presence of magnetic field

In this case, we obtain the following system of equations for the variables c_1, c_3 (let $eB \Rightarrow B$):

$$[(3+\frac{2\sigma}{\sigma+2k})(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{(m+Br^2/2+1)^2}{r^2})+(3-\frac{2\sigma}{\sigma+2k})B+2\sigma(\sigma+k)]c_1$$
$$+3[\frac{d^2}{dr^2}+\frac{1-2(m+Br^2/2)}{r}\frac{d}{dr}+\frac{(m+Br^2/2)^2-1}{r^2}-B]c_3=0,$$
$$[(3+\frac{2\sigma}{\sigma-2k})(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{(m+Br^2/2-1)^2}{r^2})+(-3+\frac{2\sigma}{\sigma-2k})B+2\sigma(\sigma-k)]c_3$$

Whence it follows 4-th order equations for c_1 and c_3:

$$c_1'''' + \frac{2}{r}c_1''' + (-\frac{e^2B^2r^2}{2}+\frac{5\sigma^3-8\sigma k^2-8\sigma mB+8\sigma B+12kB}{4\sigma}-\frac{2m^2+4m+3}{r^2})c_1''$$
$$+(-\frac{3}{2}B^2r+\frac{1}{4}\frac{5\sigma^3-8\sigma k^2-8\sigma mB+8\sigma B+12kB}{r\sigma}+\frac{2m^2+4m+3}{r^3})c_1'$$
$$+[\frac{1}{16}B^4r^4-\frac{1}{16}\frac{B^2(5\sigma^3-8\sigma k^2-8\sigma mB+8\sigma B+12kB)r^2}{\sigma}$$
$$-\frac{1}{4}\frac{(m+1)^2(5\sigma^3-8\sigma k^2-8\sigma mB+8\sigma B+12kB)}{r^2\sigma}+\frac{(m+3)(m-1)(m+1)^2}{r^4}$$
$$-\frac{1}{4\sigma}(-\sigma^5+5\sigma^3Bm-5\sigma^3B+5\sigma^3k^2-21\sigma^2kB-6\sigma m^2B^2+$$
$$+4\sigma B^2m+16B^2\sigma-8\sigma k^2Bm+8\sigma k^2B-4\sigma k^4+12kB^2m-12kB^2+12k^3B)]c_1=0;$$
$$c_3''''+\frac{2}{r}c_3'''+(-\frac{1}{2}B^2r^2+\frac{1}{4}\frac{5\sigma^3-8\sigma k^2-8\sigma mB-8\sigma B+12kB}{\sigma}-\frac{2m^2-4m+3}{r^2})c_3''$$
$$+(-\frac{3}{2}B^2r+\frac{1}{4}\frac{5\sigma^3-8\sigma k^2-8\sigma mB-8\sigma B+12kB}{r\sigma}+\frac{2m^2-4m+3}{r^3})c_3'$$
$$+[\frac{1}{16}B^4r^4-\frac{1}{16}\frac{B^2(5\sigma^3-8\sigma k^2-8\sigma mB-8\sigma B+12kB)r^2}{\sigma}-$$
$$-\frac{1}{4}\frac{(m-1)^2(5\sigma^3-8\sigma k^2-8\sigma mB-8\sigma B+12kB)}{r^2\sigma}+\frac{(m-3)(m+1)(m-1)^2}{r^4}$$
$$-\frac{1}{4\sigma}(-\sigma^5+5\sigma^3Bm+5\sigma^3B+5\sigma^3k^2-21\sigma^2kB-6\sigma m^2B^2$$
$$-4\sigma B^2m+16B^2\sigma-8\sigma k^2Bm-8\sigma k^2B-4\sigma k^4+12kB^2+12kB^2m+12k^3B)]c_3=0.$$

Both equations can be factorized. For the variable c_1 we get

$$\{\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{1}{4}B^2r^2-\frac{(m+1)^2}{r^2}$$
$$+\frac{\sigma\left(5\sigma^2-8k^2\right)-8B\sigma\left(m-1\right)+12kB\pm3\sqrt{\sigma^6-24kB\sigma^3+16\left(2\sigma^2+k^2\right)B^2}}{8\sigma}\}$$
$$\times\{\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{1}{4}B^2r^2-\frac{\left(m+1\right)^2}{r^2}$$

$$+\frac{\sigma\left(5\sigma^2-8k^2\right)-8B\sigma\left(m-1\right)+12kB\mp 3\sqrt{\sigma^6-24kB\sigma^3+16\left(2\sigma^2+k^2\right)B^2}}{8\sigma}\}c_1=0.$$

To get the equation for the variable c_3 it suffices to make one formal change $m+1\Leftrightarrow m-1$, so we omit its explicit form. It suffices to examine one 2-nd order equation

$$\{\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{1}{4}B^2r^2-\frac{(m+1)^2}{r^2}$$
$$+\frac{\sigma\left(5\sigma^2-8k^2\right)-8B\sigma\left(m-1\right)+12kB\pm 3\sqrt{\sigma^6-24kB\sigma^3+16\left(2\sigma^2+k^2\right)B^2}}{8\sigma}\}c_1=0.$$

In the variable $x=Br^2/2$, it reads

$$\frac{d^2c_1}{dx^2}+\frac{1}{x}\frac{dc_1}{dx}+[-\frac{1}{4}-\frac{1}{4}\frac{(m+1)^2}{x^2}$$
$$+\frac{\sigma\left(5\sigma^2-8k^2\right)-8B\sigma\left(m-1\right)+12kB\pm 3\sqrt{\sigma^6-24kB\sigma^3+16\left(2\sigma^2+k^2\right)B^2}}{16\sigma Bx}]c_1=0.$$

With the use of substitution $c_1(x)=e^{Ax}x^C F(x)$, we obtain the following equation for $F(x)$:

$$x\frac{d^2F}{dx^2}+\left(2Ax+2C+1\right)\frac{dF}{dx}+\{(A^2-\frac{1}{4})x+\frac{1}{4}\frac{4C^2-(m+1)^2}{x}$$
$$+\frac{1}{16}\frac{1}{\sigma B}[16A\sigma(1+2C)B+\sigma(5\sigma^2-8k^2)-8B\sigma(m-1)+12kB$$
$$\pm 3\sqrt{\sigma^6-24kB\sigma^3+16\left(2\sigma^2+k^2\right)B^2}]\}F=0.$$

Imposing restrictions on parameters, $A=-\frac{1}{2}$ and $C=+|\frac{m+1}{2}|$, we simplify the above equation

$$x\frac{d^2F}{dx^2}+\left(2C+1-x\right)\frac{dF}{dx}+\frac{1}{16}\frac{1}{\sigma B}[-8\sigma(1+2C)B+\sigma(5\sigma^2-8k^2)-8B\sigma(m-1)+12kB$$
$$\pm 3\sqrt{\sigma^6-24kB\sigma^3+16\left(2\sigma^2+k^2\right)B^2}]F=0;$$

it may be identified with the confluent hypergeometric equation with parameters

$$\alpha=-\frac{1}{16}\frac{1}{\sigma eB}[-8\sigma(1+2C)B+\sigma(5\sigma^2-8k^2)-8B\sigma(m-1)+12kB$$
$$\pm 3\sqrt{\sigma^6-24kB\sigma^3+16(2\sigma^2+k^2)B^2}],\quad \gamma=2C+1.$$

The known polynomial condition gives the following quantization rule

$$\frac{1}{16}\frac{1}{\sigma B}[-8\sigma B(1+2C)-8B\sigma(m-1)+\sigma(5\sigma^2-8k^2)+12kB$$
$$\pm 3\sqrt{\sigma^6-24kB\sigma^3+16\left(2\sigma^2+k^2\right)B^2}]=n,\quad n=0,1,2,...,$$

or differently

$$\sigma(5\sigma^2 - 8k^2) + 12kB \pm 3\sqrt{\sigma^6 - 24kB\sigma^3 + 16(2\sigma^2 + k^2)B^2} = 8\sigma B(2n + m + |m+1|). \quad (36)$$

With the use of notation $N = 2n + m + |m+1|$, we get

$$\pm 3\sqrt{\sigma^6 - 24kB\sigma^3 + 16(2\sigma^2 + k^2)B^2} = 8\sigma BN - \sigma(5\sigma^2 - 8k^2) - 12kB. \quad (37)$$

After squaring the above equation we obtain algebraic equation with the structure

$$\sigma\left[a_5\sigma^5 + a_4\sigma^4 + ... + a_0\right] = 0,$$

or explicitly

$$\sigma^6 - 5\left(BN + k^2\right)\sigma^4 + 21kB\sigma^3 + \left[4\left(BN + k^2\right)^2 - 18B^2\right]\sigma^2 - 12kB\left(BN + k^2\right)\sigma = 0. \quad (38)$$

So we arrive to 5-th order equation (let $BN + k^2 = \gamma$; the root $\sigma = 0$ is nonphysical)

$$\sigma^5 - 5\gamma\sigma^3 + 21kB\sigma^2 + \left[4\gamma^2 - 18B^2\right]\sigma - 12kB\gamma = 0. \quad (39)$$

In dimensionless variables $X = \sigma / k, b = B / k^2, \Gamma = \gamma / \sigma^2 = bN + 1,$ it reads

$$X^5 - 5\Gamma X^3 + 21bX^2 + (4\Gamma^2 - 18b^2)X - 12b\Gamma = 0; \quad (40)$$

this equation may be studied numerically.

To the spectra for σ defined by (39)–(40), there correspond the following explicit solutions

$$c_1(x) = e^{-x/2}x^C G(x), \quad C = +\frac{|m+1|}{2}. \quad (41)$$

When solving the equation for function c_3 we obtain similar result with formal change $m+1 \Leftrightarrow m-1$. In this case, we have the following solutions

$$\begin{gathered} c_3(x) = e^{-x/2}x^{C'}G(x), \quad C' = +\frac{|m-1|}{2}, \quad m' = m - 2, \\ 2n + m' + |m'+1| = N', \quad bN' + 1 = \Gamma', \\ X'^5 - 5\Gamma' X'^3 + 21bX'^2 + (4\Gamma'^2 - 18b^2)X' - 12b\Gamma' = 0. \end{gathered} \quad (42)$$

Solutions (42) and (41) provide us with one the same spectrum for σ.

Conclusions

Explicit form of the helicity operator for symmetric 2-nd rank tensor describing the spin 2 particle is specified in cylindrical coordinates. After separating the variables the system of 10 first order differential equations is derived. It is split into two independent subsystems of 4 and 6 equations. The system of 4 equations is solved straightforwardly in terms of confluent hypergeometric functions, there are found corresponding eigenvalues and eigenfunctions. Subsystem of 6 equations leads to one ordinary 4-th order differential equation. Corresponding operator is factorized into permutable 2-nd order operators, so the problem reduces to solving two differential equations of the 2-nd order. Their solutions are constructed in terms of Bessel functions. This analysis is extended to presence of external uniform magnetic field, when solutions are constructed in terms of confluent hypergeometric functions. The helicity eigenvalues are found in inexplicit form, as solutions of polynomial equations of 3-rd and 5-th orders.

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