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DIRAC – KÄHLER PARTICLE IN THE EXTERNAL MAGNETIC FIELD, CYLINDRICAL TETRAD AND FEDOROV – GRONSKIY METHOD

16-component system of equations for the Dirac – Kähler particle in the presence of the external uniform magnetic field has been studied. This equation describes a multi-spin boson field equivalent to the scalar, pseudoscalar, vector, pseudovecto, and antisymmetric tensor. On the searched solutions, we diagonalize operators of the energy, the third projection of the total angular momentum, and the third projection of the linear momentum. After separating the variables, we derive the system of sixteen first order differential equations in polar coordinate. To resolve this system, we apply the method by Fedorov – Gronskiy based on projective operators constructed from generator j^{12} for the 2-rank bispinor. According to this approach, we decompose the complete wave function into three 16-dimensional projective constituents, each expressed trough only one corresponding function of the polar coordinate. There are imposed additional differential constraints which permit us to transform all equations to algebraic form. Three basic variables are constructed in terms of the confluent hypergeometric functions, at this a quantization rule arises due to the presence of the external magnetic field. The 16-dimensional structure of solutions is determined by the linear algebraic system of equations. We have found five lineraly independent solutions.

Key words: *Dirac – Kähler particle, external magnetic field, cylindrical symmetry, tetrad formalism, projective operators, exact solutions.*

Частица Дирака – Кэлера во внешнем магнитном поле, цилиндрическая тетрада и метод Федорова – Гронского

Исследуется 16-компонентная система уравнений для частицы Дирака – Кэлера в присутствии внешнего однородного магнитного поля. Эти уравнения описывают мультиспиновое бозонное поле, эквивалентное скаляру, псевдоскалярному вектору, псевдовектору и антисимметричному тензору. На строящихся решениях диагонализируются операторы энергии, третьей проекции полного углового момента и третьей проекции импульса. После разделения переменных выведена система из 16 дифференциальных уравнений первого порядка по полярной координате. Чтобы решить эту систему, используем метод Федорова – Гронского, который основан на применении проективных операторов, строящихся из генератора j^{12} для Биспинора 2-го ранга. Согласно этому методу, полная волновая функция раскладывается в сумму трех 16-компонентных проективных составляющих, каждая из которых выражается только через одну соответствующую функцию от полярной координаты. Накладываются дополнительные дифференциальные ограничения, которые позволяют преобразовать все уравнения к алгебраическому виду. Три основные переменные найдены в терминах вырожденных гипергеометрических функций, при этом возникает правило квантования для основного спектрального параметра, обусловленное присутствием магнитного поля. 16-мерная структура решений определяется из анализа возникающей системы алгебраических уравнений. Найдены 5 линейно независимых решений с соответствующим спектром энергии частиц.

Ключевые слова: *Частица Дирака – Кэлера, внешнее магнитное поле, цилиндрическая симметрия, тетрадный формализм, проективные операторы, точные решения.*

Introduction

As the Dirac equation was proposed [1], there appeared papers in which they tried to reproduce Dirac's results within the tensor representations of the Lorentz group [2; 3]. Later, an equivalent approach within the other mathematical techniques was proposed by Kähler [4].

Scientific literature concerned with this field is enormous – see the review and references in [5–7]; also see [8–23].

Three most interesting points for this field are: in flat Minkowski space there exist tensor and spinor formulations of the theory; in the initial tensor form there are presented tensors with different intrinsic parities; there exist different views about physical interpretation of the object: whether it is a composite boson or a set of four fermions.

In this paper we will examine the problem of this particle in external uniform magnetic field. In Section 1, we specify the main equation in cylindrical coordinates and corresponding diaogonal tetrad, and then separate the variables. As a result, we derive the system of 16 first order differential equations in the polar coordinate. To resolve this system we apply the method by Fedorov – Groskiy [24], the last is based on the use of projective operators constructed with the use of generator J^{12} for the wave function with the properties of 2nd rank bispinor under the Lorentz group. Correspondingly. we present the complete wave function as the sum of three projective constituents, each of dimension 16.

According the general method [24], each projective constituent is determined trough one coresponding function of the polar coordinate, $F_1(r), F_2(r), F_3(r)$. Besides, special differential constraints consistent with the all 16 equations are introduced, they permit to transform these equations to algebraic form. In section 2, we study the introduced differential constraints for basic functions $F_1(r), F_2(r), F_3(r)$. They lead to 2nd order equations for all three functions, solutions for them may be found in terms of the confluent hypergeometric functions. In usual way, we derive the quantization rule for some spectral parameter, which later will be related with the possible energy values of the particle. In section 3, we study the 16-dimensional matrix structure of possible solutions, which are determined by the linear homogeneous algebraic system of equations. In this way, we find five linearly independent solutions for the Dirac– Kähler particle in the magnetic field.

1. Separating the variables

In Minkowski space and Cartesian coordinates, the Dirac – Kähler equation may be presented as spinor equation for a 2-rank bispinor [7]:

$$[i\,\gamma^a \partial_a - m\,]\,U(x) = 0\,, \tag{1}$$

or alternatively as the set of tensor equations for scalar, pseudoscalar, vector, pseudovector, and antisymmetric tensor [7]:

$$\begin{gathered}
\partial_l \Psi + m\Psi_l = 0\,, \qquad \partial_l \tilde{\Psi} + m\tilde{\Psi}_l = 0\,, \\
\partial_l \Psi + \partial_a \Psi_{la} - m\Psi_l = 0\,, \\
\partial_l \tilde{\Psi} - \frac{1}{2}\,\varepsilon_l^{\ amn}\,\partial_a \Psi_{mn} - m\tilde{\Psi}_l = 0\,, \\
\partial_m \Psi_n - \partial_n \Psi_m + \varepsilon_{mn}^{\ \ ab}\,\partial_a \tilde{\Psi}_b - m\Psi_{mn} = 0\,.
\end{gathered} \tag{2}$$

These two descriptions are related in the following way: let 2-nd rank bispinor be parameterized as [7]

$$U(x) = \left[-i\Psi + \gamma^l\ \Psi_l + i\sigma^{mn}\ \Psi_{mn} + \gamma^5\ \tilde{\Psi} + i\gamma^l\gamma^5\ \tilde{\Psi}_l\ \right]E^{-1}\,. \tag{3}$$

The quantity E is a bispinor metrical matrix with the following properties

$$E = \begin{vmatrix} \varepsilon & 0 \\ 0 & \dot{\varepsilon}^{-1} \end{vmatrix} = \begin{vmatrix} \varepsilon_{\alpha\beta} & 0 \\ 0 & \varepsilon^{\dot{\alpha}\dot{\beta}} \end{vmatrix} = \begin{vmatrix} i\sigma^2 & 0 \\ 0 & -i\sigma^2 \end{vmatrix}, \tag{4}$$
$$E^2 = -I\,, \qquad \tilde{E} = -E\,, \qquad Sp\ E = 0\,, \qquad \tilde{\sigma}^{ab} E = -E\sigma^{ab}\,;$$

the inverse transformation to (3) reads

$$\Psi(x) = -\frac{1}{4i}\,Sp\,[EU(x)]\,, \qquad \tilde{\Psi}(x) = \frac{1}{4}\,Sp\,[E\gamma^5 U(x)]\,,$$
$$\Psi_l(x) = \frac{1}{4}\,Sp\,[E\gamma_l U(x)]\,, \qquad \tilde{\Psi}_l(x) = \frac{1}{4i}\,Sp\,[E\gamma^5\gamma_l U(x)]\,, \tag{5}$$
$$\Psi_{mn}(x) = -\frac{1}{2i}\,Sp\,[E\sigma_{mn} U(x)]\,.$$

In cylindrical coordinates and tetrad

$$x^\alpha = (t, r, \phi, z), \quad dS^2 = dt^2 - dr^2 - r^2 d\phi^2 - dz^2,$$

the Dirac – Kähler equation [7]

$$[i\gamma^\alpha(x)(D_\alpha + \Gamma_\alpha \otimes I + I \otimes \Gamma_\alpha(x) - M]U(x) = 0, \quad D_\alpha = \partial_\alpha + ieA_\alpha, \tag{6}$$

in presence of the external uniform magnetic field $A_\phi = -\dfrac{Br^2}{2}$ (let $U(x) = V(x)/\sqrt{r}$ and $Be \to B$) takes the form

$$[i\gamma^0\partial_t + i\gamma^1\partial_r + \frac{1}{r}\gamma^2 \otimes \sigma^{12} + \frac{\gamma^2}{r}(i\partial_\phi + \frac{Br^2}{2}) + i\gamma^3\partial_z - M]V(x) = 0. \tag{7}$$

We will search for solutions in the form

$$V(t, r, \phi, z) = e^{-i\varepsilon t} e^{im\phi} e^{ikz} F(r), \quad F(r) = \begin{vmatrix} f_{11} & f_{12} & f_{13} & f_{14} \\ f_{21} & f_{22} & f_{23} & f_{24} \\ f_{31} & f_{32} & f_{33} & f_{34} \\ f_{41} & f_{42} & f_{43} & f_{44} \end{vmatrix}, \tag{8}$$

then the equation reads

$$[\varepsilon\gamma^0 + i\gamma^1\frac{d}{dr} + \frac{1}{r}\gamma^2 \otimes \sigma^{12} + \frac{\gamma^2}{r}(-m + \frac{eBr^2}{2}) - k\gamma^3 - M]F(r) = 0. \tag{9}$$

Recall expressions for Dirac matrices in spinor basis

$$\gamma^0 = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix}, \quad \gamma^1 = \begin{vmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix}, \quad \gamma^2 = \begin{vmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{vmatrix},$$

$$\gamma^3 = \begin{vmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix}, \quad \sigma^{12} = \tilde{\sigma}^{12} = \begin{vmatrix} 1/2 & 0 & 0 & 0 \\ 0 & -1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & -1/2 \end{vmatrix}.$$

Below we will apply the method by Fedorov – Gronskiy [24].

To this end, we present the complete wave function as the 16-dimensional column

$$F = \begin{vmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{vmatrix}, \quad F_1 = \begin{vmatrix} f_{11} \\ f_{21} \\ f_{31} \\ f_{41} \end{vmatrix}, F_2 = \begin{vmatrix} f_{12} \\ f_{22} \\ f_{32} \\ f_{42} \end{vmatrix}, F_3 = \begin{vmatrix} f_{13} \\ f_{23} \\ f_{33} \\ f_{43} \end{vmatrix}, F_4 = \begin{vmatrix} f_{14} \\ f_{24} \\ f_{34} \\ f_{44} \end{vmatrix}.$$

Correspondingly we should present the third projection of the spin $Y = S_3$ in 16-dimensional form

$$Y = \begin{vmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{vmatrix}, \qquad (10)$$

Let us note the eigenvalues of the matrix Y:

$$\det(Y - \lambda I_{16\times 16}) = \lambda^8 \left(\lambda^2 - 1\right)^4 = 0 \quad \Rightarrow$$
$$\lambda = -1 \quad \textit{multiplicity}\ 4; \quad \lambda = +1 \quad \textit{multiplicity}\ 4; \quad \lambda = 0 \quad \textit{multiplicity}\ 8. \qquad (11)$$

We readily verify that the minimal equation for this matrix has the form $Y^3 - Y = 0$; correspondingly, we may introduce three projective operators

$$P_1 = \frac{1}{2} Y(Y+1), \quad P_2 = \frac{1}{2} Y(Y-1), \quad P_3 = 1 - Y^2,$$

$$P_1 + P_2 + P_3 = 1, \quad P_1^2 = P_1, \quad P_2^2 = P_2, \quad P_3^2 = P_3;$$

explicit form of them is

$$P_1 = \begin{vmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 & -1/2 & 0 & 0 \\
0 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1/2 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{vmatrix},$$

$$P_2 = \begin{vmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 & -1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 \\
0 & 0 & -1/2 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{vmatrix},$$

$$
P_3 = \begin{vmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{vmatrix}.
$$

Therefore, three projective constituents are defined by the formulas (according to Fedorov – Gronskiy method [24], each projective constituent should be determined through one function):

$$
\Psi_1(r) = \begin{vmatrix}
f_{11}\\ 0\\ (f_{13}+f_{31})/2\\ 0\\ 0\\ 0\\ 0\\ (f_{42}-f_{24})/2\\ (f_{13}+f_{31})/2\\ 0\\ f_{33}\\ 0\\ 0\\ -(f_{42}-f_{24})/2\\ 0\\ 0
\end{vmatrix} F_1(r), \quad
\Psi_2(r) = \begin{vmatrix}
0\\ 0\\ -(f_{13}-f_{31})/2\\ 0\\ 0\\ f_{22}\\ 0\\ (f_{42}+f_{24})/2\\ (f_{13}-f_{31})/2\\ 0\\ 0\\ 0\\ 0\\ (f_{42}+f_{24})/2\\ 0\\ f_{44}
\end{vmatrix} F_2(r), \quad
\Psi_3(r) = \begin{vmatrix}
0\\ f_{21}\\ 0\\ f_{41}\\ f_{12}\\ 0\\ f_{32}\\ 0\\ 0\\ f_{23}\\ 0\\ f_{43}\\ f_{14}\\ 0\\ f_{34}\\ 0
\end{vmatrix} F_3(r). \qquad (12)
$$

It is convenient to introduce the following notations

$$\frac{1}{2}(f_{13}+f_{31})=A,\quad \frac{1}{2}(f_{13}-f_{31})=B,\quad \frac{1}{2}(f_{42}-f_{24})=C,\quad \frac{1}{2}(f_{42}+f_{24}=D,$$
$$f_{13}=A+B,\quad f_{31}=A-B,\quad f_{42}=D+C,\quad f_{24}=D-C, \tag{13}$$

$$\Psi_1(r)=\left|\begin{array}{c} f_{11}\\ 0\\ A\\ 0\\ 0\\ 0\\ 0\\ C\\ A\\ 0\\ f_{33}\\ 0\\ 0\\ -C\\ 0\\ 0 \end{array}\right| F_1(r),\quad \Psi_2(r)=\left|\begin{array}{c} 0\\ 0\\ -B\\ 0\\ 0\\ f_{22}\\ 0\\ D\\ B\\ 0\\ 0\\ 0\\ 0\\ D\\ 0\\ f_{44} \end{array}\right| h\, F_2(r),\quad \Psi_3(r)=\left|\begin{array}{c} 0\\ f_{21}\\ 0\\ f_{41}\\ f_{12}\\ 0\\ f_{32}\\ 0\\ 0\\ f_{23}\\ 0\\ f_{43}\\ f_{14}\\ 0\\ f_{34}\\ 0 \end{array}\right| F_3(r).$$

Now we turn to the system of equations after separating the variables (let $\mu = m-\dfrac{Br^2}{2}$):

$$(\frac{d}{dr}+\frac{\mu-1/2}{r})f_{41}-iMf_{11}-i\varepsilon f_{31}+ikf_{31}=0,(\frac{d}{dr}+\frac{\mu+1/2}{r})f_{42}-iMf_{12}-i\varepsilon f_{32}+ikf_{32}=0,$$

$$(\frac{d}{dr}+\frac{\mu-1/2}{r})f_{43}-iMf_{13}-i\varepsilon f_{33}+ikf_{33}=0,(\frac{d}{dr}+\frac{\mu+1/2}{r})f_{44}-iMf_{14}-i\varepsilon f_{34}+ikf_{34}=0,$$

$$(\frac{d}{dr}-\frac{\mu-1/2}{r})f_{31}-iMf_{21}-i\varepsilon f_{41}-ikf_{41}=0,(\frac{d}{dr}-\frac{\mu+1/2}{r})f_{32}-iMf_{22}-i\varepsilon f_{42}-ikf_{42}=0,$$

$$(\frac{d}{dr}-\frac{\mu-1/2}{r})f_{33}-iMf_{23}-i\varepsilon f_{43}-ikf_{43}=0,(\frac{d}{dr}-\frac{\mu+1/2}{r})f_{34}-iMf_{24}-i\varepsilon f_{44}-ikf_{44}=0,$$

$$-(\frac{d}{dr}+\frac{\mu-1/2}{r})f_{21}-iMf_{31}-i\varepsilon f_{11}-ikf_{11}=0,-(\frac{d}{dr}+\frac{\mu+1/2}{r})f_{22}-iMf_{32}-i\varepsilon f_{12}-ikf_{12}=0,$$

$$-(\frac{d}{dr}+\frac{\mu-1/2}{r})f_{23}-iMf_{33}-i\varepsilon f_{13}-ikf_{13}=0,-(\frac{d}{dr}+\frac{\mu+1/2}{r})f_{24}-iMf_{34}-i\varepsilon f_{14}-ikf_{14}=0,$$

$$-(\frac{d}{dr}-\frac{\mu-1/2}{r})f_{11}-iMf_{41}-i\varepsilon f_{21}+ikf_{21}=0,-(\frac{d}{dr}-\frac{\mu+1/2}{r})f_{12}-iMf_{42}-i\varepsilon f_{22}+ikf_{22}=0,$$

$$-(\frac{d}{dr}-\frac{\mu-1/2}{r})f_{13}-iMf_{43}-i\varepsilon f_{23}+ikf_{23}=0,-(\frac{d}{dr}-\frac{\mu+1/2}{r})f_{14}-iMf_{44}-i\varepsilon f_{24}+ikf_{24}=0.$$

With the shortening notations

$$\frac{d}{dr}+\frac{\mu+1/2}{r}=a_{\mu+1/2}, \quad \frac{d}{dr}+\frac{\mu-1/2}{r}=a_{\mu-1/2},$$
$$\frac{d}{dr}-\frac{\mu+1/2}{r}=b_{\mu+1/2}, \quad \frac{d}{dr}-\frac{\mu-1/2}{r}=b_{\mu-1/2}, \tag{14}$$

the above equations read

$$a_{\mu-1/2}f_{41}-iMf_{11}+(-i\varepsilon+ik)f_{31}=0, \quad a_{\mu+1/2}f_{42}-iMf_{12}+(-i\varepsilon+ik)f_{32}=0,$$
$$a_{\mu-1/2}f_{43}-iMf_{13}+(-i\varepsilon+ik)f_{33}=0, \quad a_{\mu+1/2}f_{44}-iMf_{14}+(-i\varepsilon+ik)f_{34}=0,$$
$$b_{\mu-1/2}f_{31}-iMf_{21}+(-i\varepsilon-ik)f_{41}=0, \quad b_{\mu+1/2}f_{32}-iMf_{22}+(-i\varepsilon-ik)f_{42}=0,$$
$$b_{\mu-1/2}f_{33}-iMf_{23}+(-i\varepsilon-ik)f_{43}=0, \quad b_{\mu+1/2}f_{34}-iMf_{24}+(-i\varepsilon-ik)f_{44}=0,$$
$$-a_{\mu-1/2}f_{21}-iMf_{31}+(-i\varepsilon-ik)f_{11}=0, \quad -a_{\mu+1/2}f_{22}-iMf_{32}+(-i\varepsilon-ik)f_{12}=0,$$
$$-a_{\mu-1/2}f_{23}-iMf_{33}+(-i\varepsilon-ik)f_{13}=0, \quad -a_{\mu+1/2}f_{24}-iMf_{34}+(-i\varepsilon-ik)f_{14}=0,$$
$$-b_{\mu-1/2}f_{11}-iMf_{41}+(-i\varepsilon+ik)f_{21}=0, \quad -b_{\mu+1/2}f_{12}-iMf_{42}+(-i\varepsilon+ik)f_{22}=0,$$
$$-b_{\mu-1/2}f_{13}-iMf_{43}+(-i\varepsilon+ik)f_{23}=0, \quad -b_{\mu+1/2}f_{14}-iMf_{44}+(-i\varepsilon+ik)f_{24}=0.$$

Let us transform these equations to other variables

$$f_{13}=(A+B), \quad f_{31}=(A-B), \qquad f_{42}=(D+C), \quad f_{24}=(D-C); \tag{15}$$

then we get

$$
\begin{array}{ll}
1 & a_{\mu-1/2}f_{41}-iMf_{11}+(-i\varepsilon+ik)(A-B)=0,\\
2 & a_{\mu+1/2}(D+C)-iMf_{12}+(-i\varepsilon+ik)f_{32}=0,\\
3 & a_{\mu-1/2}f_{43}-iM(A+B)+(-i\varepsilon+ik)f_{33}=0,\\
4 & a_{\mu+1/2}f_{44}-iMf_{14}+(-i\varepsilon+ik)f_{34}=0,\\
5 & b_{\mu-1/2}(A-B)-iMf_{21}+(-i\varepsilon-ik)f_{41}=0,\\
6 & b_{\mu+1/2}f_{32}-iMf_{22}+(-i\varepsilon-ik)(D+C)=0,\\
7 & b_{\mu-1/2}f_{33}-iMf_{23}+(-i\varepsilon-ik)f_{43}=0,\\
8 & b_{\mu+1/2}f_{34}-iM(D-C)+(-i\varepsilon-ik)f_{44}=0,\\
9 & -a_{\mu-1/2}f_{21}-iM(A-B)+(-i\varepsilon-ik)f_{11}=0,\\
10 & -a_{\mu+1/2}f_{22}-iMf_{32}+(-i\varepsilon-ik)f_{12}=0,\\
11 & -a_{\mu-1/2}f_{23}-iMf_{33}+(-i\varepsilon-ik)(A+B)=0,\\
12 & -a_{\mu+1/2}(D-C)-iMf_{34}+(-i\varepsilon-ik)f_{14}=0,\\
13 & -b_{\mu-1/2}f_{11}-iMf_{41}+(-i\varepsilon+ik)f_{21}=0,\\
14 & -b_{\mu+1/2}f_{12}-iM(D+C)+(-i\varepsilon+ik)f_{22}=0,\\
15 & -b_{\mu-1/2}(A+B)-iMf_{43}+(-i\varepsilon+ik)f_{23}=0,\\
16 & -b_{\mu+1/2}f_{14}-iMf_{44}+(-i\varepsilon+ik)(D-C)=0.
\end{array}
\tag{16}
$$

It is convenient to apply (temporally) the shortening notations

$$a_{\mu-1/2} = a_1, \quad a_{\mu+1/2} = a_2, \quad b_{\mu-1/2} = b_1, \quad b_{\mu+1/2} = b_2;$$

further we derive

2) + 12)

$$a_2(D+C) - iMf_{12} + i(-\varepsilon+k) f_{32} - a_2(D-C) - iMf_{34} + i(-\varepsilon-k) f_{14} = 0,$$

2) − 12)

$$a_2(D+C) - iMf_{12} + i(-\varepsilon+k) f_{32} + a_2(D-C) + iMf_{34} - i(-\varepsilon-k) f_{14} = 0;$$

3) + 9)

$$a_1 f_{43} + i(-\varepsilon+k) f_{33} - a_1 f_{21} + i(-\varepsilon-k) f_{11} - iM(A+B) - iM(A-B) = 0,$$

3) − 9)

$$a_1 f_{43} + i(-\varepsilon+k) f_{33} + a_1 f_{21} - i(-\varepsilon-k) f_{11} - iM(A+B) + iM(A-B) = 0;$$

5) + 15)

$$b_1(A-B) - iMf_{21} + i(-\varepsilon-k) f_{41} - b_1(A+B) - iMf_{43} + i(-\varepsilon+k) f_{23} = 0,$$

5) − 15)

$$b_1(A-B) - iMf_{21} + i(-\varepsilon-k) f_{41} + b_1(A+B) + iMf_{43} - i(-\varepsilon+k) f_{23} = 0;$$

8) + 14)

$$b_2 f_{34} + i(-\varepsilon-k) f_{44} - b_2 f_{12} + i(-\varepsilon+k) f_{22} - iM(D-C) - iM(D+C) = 0,$$

8) − 14)

$$b_2 f_{34} + i(-\varepsilon-k) f_{44} + b_2 f_{12} - i(-\varepsilon+k) f_{22} - iM(D-C) + iM(D+C) = 0;$$

equations 1) and 11)

$$1) \qquad a_1 f_{41} + i(-\varepsilon+k)(A-B) - iMf_{11} = 0,$$

$$11) \qquad -a_1 f_{23} + i(-\varepsilon-k)(A+B) - iMf_{33} = 0,$$

give

$$A = \frac{ia_1\left(f_{23}(k+\varepsilon) + f_{41}(k-\varepsilon)\right) + f_{11}M(k-\varepsilon) - f_{33}M(k+\varepsilon)}{2\left(k^2-\varepsilon^2\right)},$$

$$B = \frac{ia_1\left(f_{23}(k+\varepsilon) + f_{41}(\varepsilon-k)\right) + f_{11}M(\varepsilon-k) - f_{33}M(k+\varepsilon)}{2(k-\varepsilon)(k+\varepsilon)};$$

equations 6) and 16)

$$6) \qquad b_2 f_{32} + i(-\varepsilon-k)(D+C) - iMf_{22} = 0,$$

$$16) \qquad -b_2 f_{14} + i(-\varepsilon+k)(D-C) - iMf_{44} = 0,$$

give

$$C = \frac{-M\left(f_{22}(k+\varepsilon) + f_{44}(k-\varepsilon)\right) + ib_2\left(f_{14}(k-\varepsilon) - f_{32}(k+\varepsilon)\right)}{2\left(k^2-\varepsilon^2\right)},$$

$$D=-\frac{ib_2\left(f_{14}(k-\varepsilon)+f_{32}(k+\varepsilon)\right)+f_{22}M(k+\varepsilon)+f_{44}M(\varepsilon-k)}{2(k-\varepsilon)(k+\varepsilon)};$$

the remaining equations preserve their form

$$4)\qquad a_2 f_{44}+i(-\varepsilon+k)f_{34}-iMf_{14}=0,$$

$$7)\qquad b_1 f_{33}+i(-\varepsilon-k)f_{43}-iMf_{23}=0,$$

$$10\qquad -a_2 f_{22}+i(-\varepsilon-k)f_{12}-iMf_{32}=0,$$

$$13)\qquad -b_1 f_{11}+i(-\varepsilon+k)f_{21}-iMf_{41}=0.$$

Recall dependence of 16variables on the three basic functions

$$\begin{gathered}\Psi_1,\ \ f_{11}, f_{33}, A, C \Rightarrow F_1; \qquad \Psi_2,\ \ f_{22}, f_{44}, B, D \Rightarrow F_2;\\ \Psi_3,\ \ f_{12}, f_{21},\ f_{14}, f_{41},\ f_{23}, f_{32},\ f_{34}, f_{43} \Rightarrow \quad F_3.\end{gathered} \qquad (17)$$

convenient to collect equations in three groups (follow the variables at the mass parameter M):

I

$$-ia_{\mu-1/2}f_{23}F_3(k+\varepsilon)-ia_{\mu-1/2}f_{41}F_3(k-\varepsilon)+2AF_1(k^2-\varepsilon^2)+f_{11}F_1M(\varepsilon-k)+f_{33}F_1M(k+\varepsilon)=0, a_{\mu-1/2}F_3=C_1F_1,$$

$$-ia_{\mu-1/2}f_{23}F_3(k+\varepsilon)+ia_{\mu-1/2}f_{41}F_3(k-\varepsilon)+2BF_2(k-\varepsilon)(k+\varepsilon)+f_{11}F_1M(k-\varepsilon)+f_{33}F_1M(k+\varepsilon)=0; \quad B=0,$$

$$-a_{\mu-1/2}f_{21}F_3+a_{\mu-1/2}f_{43}F_3-2iAF_1M-if_{11}F_1(k-\varepsilon)+if_{33}F_1(k+\varepsilon)=0,$$

$$b_{\mu+1/2}f_{12}F_3+b_{\mu+1/2}f_{34}F_3-if_{22}F_2(k+\varepsilon)+if_{44}F_2(\varepsilon-k)+2iF_1CM=0, \quad C=0, \quad b_{\mu+1/2}F_3=C_2F_2,$$

II

$$a_{\mu-1/2}f_{21}F_3+a_{\mu-1/2}f_{43}F_3-2iBF_2M+if_{11}F_1(k-\varepsilon)+if_{33}F_1(k+\varepsilon)=0, \quad B=0,$$

$$-ib_{\mu+1/2}f_{14}F_3(k-\varepsilon)+ib_{\mu+1/2}f_{32}F_3(k+\varepsilon)+f_{22}F_2M(k+\varepsilon)+f_{44}F_2M(k-\varepsilon)+2F_1C(k^2-\varepsilon^2)=0, C=0,$$

$$-b_{\mu+1/2}f_{12}F_3+b_{\mu+1/2}f_{34}F_3+if_{22}F_2(k+\varepsilon)+if_{44}F_2(\varepsilon-k)-2iF_2MD=0,$$

$$-ib_{\mu+1/2}f_{14}F_3(k-\varepsilon)-ib_{\mu+1/2}f_{32}F_3(k+\varepsilon)-f_{22}F_2M(k+\varepsilon)+f_{44}F_2M(k-\varepsilon)+2F_2D(k^2-\varepsilon^2)=0;$$

III

$$-2b_{\mu-1/2}BF_2+if_{23}F_3(k+\varepsilon)+if_{41}F_3(\varepsilon-k)-if_{21}F_3M-if_{43}F_3M=0, \quad B=0,$$

$$-b_{\mu-1/2}f_{11}F_1+if_{21}F_3(k+\varepsilon)-if_{41}F_3M=0,$$

$$2a_{\mu+1/2}F_1C-if_{14}F_3(k-\varepsilon)+if_{32}F_3(k+\varepsilon)-if_{12}F_3M-if_{34}F_3M=0, \quad C=0,$$

$$-a_{\mu+1/2}f_{22}F_2+if_{12}F_3(\varepsilon-k)-if_{32}F_3M=0,$$

$$b_{\mu-1/2}f_{33}F_1+if_{43}F_3(\varepsilon-k)-if_{23}F_3M=0,$$

$$2Ab_{\mu-1/2}F_1-if_{23}F_3(k+\varepsilon)-if_{41}F_3(k-\varepsilon)-if_{21}F_3M+if_{43}F_3M=0,$$

$$a_{\mu+1/2} f_{44} F_2 + i f_{34} F_3 (k+\varepsilon) - i f_{14} F_3 M = 0,$$

$$2 a_{\mu+1/2} F_2 D + i f_{14} F_3 (k-\varepsilon) + i f_{32} F_3 (k+\varepsilon) - i f_{12} F_3 M + i f_{34} F_3 M = 0.$$

In accordance with the general method by Fedorov – Gronskiy [24], we should impose the relevant differential constraints which will permit us to transform all equations to algebraic form

$$B = 0, C = 0, \quad a_{\mu-1/2} F_3 = C_1 F_1, b_{\mu+1/2} F_3 = C_2 F_2, b_{\mu-1/2} F_1 = C_3 F_3, a_{\mu+1/2} F_2 = C_4 F_3; \tag{18}$$

in this way we obtain

I

$$-i f_{23} C_1 F_1 (k+\varepsilon) - i f_{41} C_1 F_1 (k-\varepsilon) + 2 A F_1 (k^2 - \varepsilon^2) + f_{11} F_1 M (\varepsilon - k) + f_{33} F_1 M (k+\varepsilon) = 0,$$

$$-i f_{23} C_1 F_1 (k+\varepsilon) + i f_{41} C_1 F_1 (k-\varepsilon) + f_{11} F_1 M (k-\varepsilon) + f_{33} F_1 M (k+\varepsilon) = 0;$$

$$- f_{21} C_1 F_1 + f_{43} C_1 F_1 - 2 i A F_1 M - i f_{11} F_1 (k-\varepsilon) + i f_{33} F_1 (k+\varepsilon) = 0,$$

$$f_{12} C_2 F_2 + f_{34} C_2 F_2 - i f_{22} F_2 (k+\varepsilon) + i f_{44} F_2 (\varepsilon - k) = 0,$$

II

$$f_{21} C_1 F_1 + f_{43} C_1 F_1 + i f_{11} F_1 (k-\varepsilon) + i f_{33} F_1 (k+\varepsilon) = 0,$$

$$-i f_{14} C_2 F_2 (k-\varepsilon) + i f_{32} C_2 F_2 (k+\varepsilon) + f_{22} F_2 M (k+\varepsilon) + f_{44} F_2 M (k-\varepsilon) = 0,$$

$$- f_{12} C_2 F_2 + f_{34} C_2 F_2 + i f_{22} F_2 (k+\varepsilon) + i f_{44} F_2 (\varepsilon - k) - 2 i F_2 M D = 0,$$

$$-i f_{14} C_2 F_2 (k-\varepsilon) - i f_{32} C_2 F_2 (k+\varepsilon) - f_{22} F_2 M (k+\varepsilon) + f_{44} F_2 M (k-\varepsilon) + 2 F_2 D (k^2 - \varepsilon^2) = 0;$$

III

$$i f_{23} F_3 (k+\varepsilon) + i f_{41} F_3 (\varepsilon - k) - i f_{21} F_3 M - i f_{43} F_3 M = 0,$$

$$- f_{11} C_3 F_3 + i f_{21} F_3 (k+\varepsilon) - i f_{41} F_3 M = 0,$$

$$-i f_{14} F_3 (k-\varepsilon) + i f_{32} F_3 (k+\varepsilon) - i f_{12} F_3 M - i f_{34} F_3 M = 0,$$

$$-C_4 F_3 f_{22} + i f_{12} F_3 (\varepsilon - k) - i f_{32} F_3 M = 0,$$

$$f_{33} C_3 F_3 + i f_{43} F_3 (\varepsilon - k) - i f_{23} F_3 M = 0,$$

$$2 A C_3 F_3 - i f_{23} F_3 (k+\varepsilon) - i f_{41} F_3 (k-\varepsilon) - i f_{21} F_3 M + i f_{43} F_3 M = 0,$$

$$f_{44} C_4 F_3 + i f_{34} F_3 (k+\varepsilon) - i f_{14} F_3 M = 0,$$

$$2 C_4 F_3 D + i f_{14} F_3 (k-\varepsilon) + i f_{32} F_3 (k+\varepsilon) - i f_{12} F_3 M + i f_{34} F_3 M = 0;$$

reducing the total multipliers in each equation we arrive at the algebraic system

I

$$-i f_{23} C_1 (k+\varepsilon) - i f_{41} C_1 (k-\varepsilon) + 2 A (k^2 - \varepsilon^2) + f_{11} M (\varepsilon - k) + f_{33} M (k+\varepsilon) = 0,$$

$$-i f_{23} C_1 (k+\varepsilon) + i f_{41} C_1 (k-\varepsilon) + f_{11} M (k-\varepsilon) + f_{33} M (k+\varepsilon) = 0;$$

$$-f_{21}C_1+f_{43}C_1-2iAM-if_{11}(k-\varepsilon)+if_{33}(k+\varepsilon)=0,$$

$$f_{12}C_2+f_{34}C_2-if_{22}(k+\varepsilon)+if_{44}(\varepsilon-k)=0,$$

II

$$f_{21}C_1+f_{43}C_1+if_{11}(k-\varepsilon)+if_{33}(k+\varepsilon)=0,$$

$$-if_{14}C_2(k-\varepsilon)+if_{32}C_2(k+\varepsilon)+f_{22}M(k+\varepsilon)+f_{44}M(k-\varepsilon)=0,$$

$$-f_{12}C_2+f_{34}C_2+if_{22}(k+\varepsilon)+if_{44}(\varepsilon-k)-2iMD=0,$$

$$-if_{14}C_2(k-\varepsilon)-if_{32}C_2(k+\varepsilon)-f_{22}M(k+\varepsilon)+f_{44}M(k-\varepsilon)+2D(k^2-\varepsilon^2)=0;$$

III

$$if_{23}(k+\varepsilon)+if_{41}(\varepsilon-k)-if_{21}M-if_{43}M=0,$$

$$-f_{11}C_3+if_{21}(k+\varepsilon)-if_{41}M=0,$$

$$-if_{14}(k-\varepsilon)+if_{32}(k+\varepsilon)-if_{12}M-if_{34}M=0,$$

$$-C_4f_{22}+if_{12}(\varepsilon-k)-if_{32}M=0,$$

$$f_{33}C_3+if_{43}(\varepsilon-k)-if_{23}M=0,$$

$$2AC_3-if_{23}(k+\varepsilon)-if_{41}(k-\varepsilon)-if_{21}M+if_{43}M=0,$$

$$f_{44}C_4+if_{34}(k+\varepsilon)-if_{14}M=0,$$

$$2C_4D+if_{14}(k-\varepsilon)+if_{32}(k+\varepsilon)-if_{12}M+if_{34}M=0.$$

2. Differential constraints

Let us study the above differential relations

$$\begin{array}{l} a_{\mu-1/2}F_3=C_1F_1,\quad b_{\mu-1/2}F_1=C_3F_3,\ let\ C_3=C_1,\\ b_{\mu+1/2}F_3=C_2F_2,\quad a_{\mu+1/2}F_2=C_4F_3,\ let\ C_4=C_2. \end{array} \tag{19}$$

We readily derive the 2nd order equations

$$\begin{array}{l} (a_{\mu-1/2}b_{\mu-1/2}-C_1^2)F_1=0,\quad (b_{\mu-1/2}a_{\mu-1/2}-C_1^2)F_3=0,\\ (b_{\mu+1/2}a_{\mu+1/2}-C_2^2)F_2=0,\quad (a_{\mu+1/2}b_{\mu+1/2}-C_2^2)F_3=0. \end{array} \tag{20}$$

Bearing in mind the definitions

$$\frac{d}{dr}+\frac{\mu+1/2}{r}=a_{\mu+1/2},\quad \frac{d}{dr}+\frac{\mu-1/2}{r}=a_{\mu-1/2},$$

$$\frac{d}{dr}-\frac{\mu+1/2}{r}=b_{\mu+1/2},\quad \frac{d}{dr}-\frac{\mu-1/2}{r}=b_{\mu-1/2},$$

we can find the needed products of operators

$$a_{\mu-1/2}b_{\mu-1/2}=\frac{d^2}{dr^2}-\frac{(4(\mu-2)\mu+3)}{4r^2},\quad b_{\mu-1/2}a_{\mu-1/2}=\frac{d^2}{dr^2}+\frac{\left(1-4\mu^2\right)}{4r^2},$$

$$b_{\mu+1/2}a_{\mu+1/2} = \frac{d^2}{dr^2} - \frac{(4\mu(\mu+2)+3)}{4r^2}, \quad a_{\mu+1/2}b_{\mu+1/2} = \frac{d^2}{dr^2} + \frac{\left(1-4\mu^2\right)}{4r^2}.$$

So we arrive at four differential equations for 3 functions:

$$(\frac{d^2}{dr^2} - \frac{(4(\mu-2)\mu+3)}{4r^2} - C_1^2)F_1 = 0, \quad (\frac{d^2}{dr^2} + \frac{\left(1-4\mu^2\right)}{4r^2} - C_1^2)F_3 = 0,$$

$$(\frac{d^2}{dr^2} - \frac{(4\mu(\mu+2)+3)}{4r^2} - C_2^2)F_2 = 0, \quad (\frac{d^2}{dr^2} + \frac{\left(1-4\mu^2\right)}{4r^2} - C_2^2)F_3 = 0.$$

The second and fourth equations agree only if $C_1^2 = C_2^2 = C^2$.
Thus, we have only three different equations:

$$\begin{aligned}
&(\frac{d^2}{dr^2} - \frac{(4\mu(\mu-2)+3)}{4r^2} - C^2)F_1 = 0,\\
&(\frac{d^2}{dr^2} - \frac{(4\mu(\mu+2)+3)}{4r^2} - C^2)F_2 = 0,\\
&(\frac{d^2}{dr^2} + \frac{\left(1-4\mu^2\right)}{4r^2} - C^2)F_3 = 0.
\end{aligned} \tag{21}$$

Allowing for that $\mu = m + Br^2/2$, we arrive at

$$\begin{aligned}
&(\frac{d^2}{dr^2} - \frac{1}{4}B^2r^2 - Bm + B - \frac{m^2}{r^2} + \frac{2m}{r^2} - \frac{3}{4r^2} - C^2)F_1 = 0,\\
&(\frac{d^2}{dr^2} - \frac{1}{4}B^2r^2 - Bm - B - \frac{m^2}{r^2} - \frac{2m}{r^2} - \frac{3}{4r^2} - C^2)F_2 = 0,\\
&(\frac{d^2}{dr^2} - \frac{1}{4}B^2r^2 - Bm - \frac{m^2}{r^2} + \frac{1}{4r^2} - C^2)F_3 = 0.
\end{aligned} \tag{22}$$

In a new variable $x = r^2$, the above equations read

$$\begin{aligned}
&(4x\frac{d^2}{dx^2} + 2\frac{d}{dx} - \frac{1}{4}B^2x - Bm + B - \frac{m^2}{x} + \frac{2m}{x} - \frac{3}{4x} - C^2)F_1 = 0,\\
&(4x\frac{d^2}{dx^2} + 2\frac{d}{dx} - \frac{1}{4}B^2x - Bm - B - \frac{m^2}{x} - \frac{2m}{x} - \frac{3}{4x} - C^2)F_2 = 0,\\
&(4x\frac{d^2}{dx^2} + 2\frac{d}{dx} - \frac{1}{4}B^2x - Bm - \frac{m^2}{x} + \frac{1}{4x} - C^2)F_3 = 0.
\end{aligned} \tag{23}$$

We use the needed substitutions for three functions:

$$(4x\frac{d^2}{dx^2}+2\frac{d}{dx}-\frac{1}{4}B^2x-Bm+B-\frac{m^2}{x}+\frac{2m}{x}-\frac{3}{4x}-C^2)x^{A_1}e^{Dx}f_1(x)=0,$$
$$(4x\frac{d^2}{dx^2}+2\frac{d}{dx}-\frac{1}{4}B^2x-Bm-B-\frac{m^2}{x}-\frac{2m}{x}-\frac{3}{4x}-C^2)x^{A_2}e^{Dx}f_2(x)=0,$$
$$(4x\frac{d^2}{dx^2}+2\frac{d}{dx}-\frac{1}{4}B^2x-Bm-\frac{m^2}{x}+\frac{1}{4x}-C^2)x^{A_3}e^{Dx}f_3(x)=0.$$

this results in

$$(\frac{8A_1D-Bm+B-C^2+2D}{x}+\frac{4A_1^2-2A_1-(m-2)m-\frac{3}{4}}{x^2}-\frac{B^2}{4}+4D^2)f_1 +$$
$$+(\frac{8A_1+2}{x}+8D)f_1'+4f_1''=0,$$
$$(\frac{8A_2D-B(m+1)-C^2+2D}{x}-\frac{(-4A_2+2m+3)(4A_2+2m+1)}{4x^2}-\frac{B^2}{4}+4D^2)f_2 +$$
$$+(\frac{8A_2+2}{x}+8D)f_2'+4f_2''=0,$$
$$(\frac{8A_3D-Bm-C^2+2D}{x}+\frac{4A_3^2-2A_3-m^2+\frac{1}{4}}{x^2}-\frac{B^2}{4}+4D^2)f_3+(\frac{8A_3+2}{x}+8D)f_3'+4f_3''=0,$$

Let us impose the evident constraints

$$-\frac{B^2}{4}+4D^2=0,\quad \frac{4A_1^2-2A_1-(m-2)m-\frac{3}{4}}{x^2}=0,$$
$$-\frac{B^2}{4}+4D^2=0,\quad \frac{(-4A_2+2m+3)(4A_2+2m+1)}{4x^2}=0,$$
$$-\frac{B^2}{4}+4D^2=0,\quad \frac{4A_3^2-2A_3-m^2+\frac{1}{4}}{x^2}=0,$$

their solutions have the form

$$B>0,\quad D=-\frac{B}{4},$$

$$A_1=\frac{1}{4}(-2m+3)>0,\quad m=0,-1,-2,...;\qquad A_1=\frac{1}{4}(2m-1)>0,\quad m=1,2,3,...;$$
$$A_2=\frac{1}{4}(-2m-1)>0,\quad m=-1,-2,-3,...;\qquad A_2=\frac{1}{4}(2m+3>0),\quad m=-1,0,+1.,...; \tag{24}$$
$$A_3=\frac{1}{4}(-2m+1)>0,\quad m=0,-1,-2,...;\qquad A_3=\frac{1}{4}(2m+1)>0,\quad m=0,+1,+2,...$$

Taking into account these conditions, we get more simple equations

$$4xf_{1''} + \left(8A_1 + 2 - 2Bx\right) f_{1'} + (-2BA_1 - B(m-1) - C^2 - \frac{B}{2}) f_1 = 0,$$

$$4xf_{2''} + \left(8A_2 + 2 - 2Bx\right) f_{2'} + (-2BA_2 - B(m+1) - C^2 - \frac{B}{2}) f_2 = 0,$$

$$4xf_{3''} + \left(8A_3 + 2 - 2Bx\right) f_{3'} + (-2BA_3 - Bm - C^2 - \frac{B}{2}) f_3 = 0,$$

In the variable $y = ax, a = \frac{B}{2}$, they take the form

$$y\frac{d^2}{dy^2} f_1 + (2A_1 + \frac{1}{4} - y)\frac{d}{dy} f_1 - (A_1 + \frac{m-1}{2} + \frac{C^2}{2B} + \frac{1}{4}) f_1 = 0,$$

$$y\frac{d^2}{dy^2} f_2 + \left(2A_2 + \frac{1}{4} - y\right)\frac{d}{dy} f_2 - (A_2 + \frac{m+1}{2} + \frac{C^2}{2B} + \frac{1}{4}) f_2 = 0,$$

$$y\frac{d^2}{dy^2} f_3 + \left(2A_3 + \frac{1}{4} - y\right)\frac{d}{dy} f_3 - (A_3 + \frac{m}{2} + \frac{C^2}{2B} + \frac{1}{4}) f_3 = 0.$$

Here we have three confluent hypergeometric equations,

$$y\frac{d^2}{dy^2} f_i + \left(c_i - y\right)\frac{d}{dy} f_1 - a_i f_1 = 0,$$

the quantization condition is $a_i = -n_i, n_i \in \{0,1,2,...\}$; whence we derive (recalling $x = r^2, y = Br^2/2$)

$$f_1,\quad C^2 = -2B(n_1 + A_1 + \frac{m-1}{2} + \frac{1}{4}),\quad F_1 = x^{A_1} e^{-Bx/4} F(2A_1 + \frac{1}{4}, -n_1, y);$$

$$f_2,\quad C^2 = -2B(n_2 + A_2 + \frac{m+1}{2} + \frac{1}{4}),\quad F_2 = x^{A_2} e^{-Bx/4} F(2A_2 + \frac{1}{4}, -n_2, y); \qquad (25)$$

$$f_3,\quad C^2 = -2B(n_3 + A_3 + \frac{m}{2} + \frac{1}{4}),\quad F_3 = x^{A_3} e^{-Bx/4} F(2A_3 + \frac{1}{4}, -n_3, y);$$

evidently, the parameter C is imaginary. Besides, the values for A_1, A_2, A_3 should be positive; depending on the sign of the quantum number m, we have two different possibilities in each case.

Depending on the values of the quantum number m, we have the following quantization rules

$$F_1,\qquad m = 0, -1, -2, ...;\quad A_1 = \frac{-m + 3/2}{2},$$

$$C^2 = -2B(n_1 + A_1 + \frac{m-1}{2} + \frac{1}{4}) = -2B(n_1 + \frac{-m+3/2}{2} + \frac{m-1}{2} + \frac{1}{4}) = -2B(n_1 + \frac{1}{2}),$$

$$F_2,\qquad m = -1, -2, -3, ...;\quad A_2 = \frac{-m - 1/2}{2},$$

$$C^2 = -2B(n_2 + A_2 + \frac{m+1}{2} + \frac{1}{4}) = -2B(n_2 + \frac{-m-1/2}{2} + \frac{m+1}{2} + \frac{1}{4}) = -2B(n_2 + \frac{1}{2}),$$

$$F_3,\qquad m=-1,-2,...;\quad A_3=\frac{-m+1/2}{2},$$

$$C^2=-2B(n_3+A_3+\frac{m}{2}+\frac{1}{4})=-2B(n_3+\frac{-m+1/2}{2}+\frac{m}{2}+\frac{1}{4})=-2B(n_3+\frac{1}{2});$$

and

$$F_1,\qquad m=+1,+2,+3,...;\quad A_1=\frac{m-1/2}{2},$$

$$C^2=-2B(n_1+A_1+\frac{m-1}{2}+\frac{1}{4})=-2B(n_1+\frac{m-1/2}{2}+\frac{m-1}{2}+\frac{1}{4})=-2B(n_1+m-\frac{1}{2});$$

$$F_2,\qquad m=0,+1,+2,...;\quad A_2=\frac{m+3/2}{2},$$

$$C^2=-2B(n_2+A_2+\frac{m+1}{2}+\frac{1}{4})=-2B(n_2+\frac{m+3/2}{2}+\frac{m+1}{2}+\frac{1}{4})=-2B(n_2+m+\frac{3}{2});$$

$$F_3,\qquad m=0,+1,+2,...;\quad A_3=\frac{m+1/2}{2},$$

$$C^2=-2B(n_3+A_3+\frac{m}{2}+\frac{1}{4})=-2B(n_3+\frac{m+1/2}{2}+\frac{m}{2}+\frac{1}{4})=-2B(n_3+m+\frac{1}{2}).$$

3. Solutions of the algebraic equations

Let us turn to solving the algebraic system, where $C_1=C_2=C_3=C$.

It is convenient to apply the matrix presentation.

We can note that the number of equations, 16, does not coincide with the number of the variables, 14.

First, consider three equations that contain the variable A:

$$-if_{23}C(k+\varepsilon)-if_{41}C(k-\varepsilon)+2A(k^2-\varepsilon^2)+f_{11}M(\varepsilon-k)+f_{33}M(k+\varepsilon)=0,$$

$$-f_{21}C+f_{43}C-2iAM-if_{11}(k-\varepsilon)+if_{33}(k+\varepsilon)=0,$$

$$2AC-if_{23}(k+\varepsilon)-if_{41}(k-\varepsilon)-if_{21}M+if_{43}M=0,$$

Expressing from the third equation the A, and substituting it two remaining we obtain

$$-\frac{if_{23}(k+\varepsilon)\left(C^2-k^2+\varepsilon^2\right)}{C}-\frac{if_{41}(k-\varepsilon)\left(C^2-k^2+\varepsilon^2\right)}{C}$$

$$+\frac{if_{21}M(k-\varepsilon)(k+\varepsilon)}{C}-\frac{if_{43}M(k-\varepsilon)(k+\varepsilon)}{C}+f_{11}M(\varepsilon-k)+f_{33}M(k+\varepsilon)=0,$$

$$\frac{f_{23}M(k+\varepsilon)}{C}+\frac{f_{41}M(k-\varepsilon)}{C}+f_{21}(\frac{M^2}{C}-C)+f_{43}(C-\frac{M^2}{C})-if_{11}(k-\varepsilon)+if_{33}(k+\varepsilon)=0,$$

so instead of three equations we get two ones.

Now, let us write down three equations that contain the variable D:

$$-f_{12}C+f_{34}C+if_{22}(k+\varepsilon)+if_{44}(\varepsilon-k)-2iMD=0,$$

$$-if_{14}C(k-\varepsilon)-if_{32}C(k+\varepsilon)-f_{22}M(k+\varepsilon)+f_{44}M(k-\varepsilon)+2D(k^2-\varepsilon^2)=0;$$

$$2CD+if_{14}(k-\varepsilon)+if_{32}(k+\varepsilon)-if_{12}M+if_{34}M=0.$$

Expressing the D from the third equations and substituting in two remaining, we get

$$\frac{f_{14}M(\varepsilon-k)}{C}-\frac{f_{32}M(k+\varepsilon)}{C}+f_{12}(\frac{M^2}{C}-C)+f_{34}(C-\frac{M^2}{C})+if_{22}(k+\varepsilon)+if_{44}(\varepsilon-k)=0,$$

$$-\frac{if_{14}(k-\varepsilon)\left(C^2+k^2-\varepsilon^2\right)}{C}-\frac{if_{32}(k+\varepsilon)\left(C^2+k^2-\varepsilon^2\right)}{C}+$$

$$+\frac{if_{12}M(k-\varepsilon)(k+\varepsilon)}{C}-\frac{if_{34}M(k-\varepsilon)(k+\varepsilon)}{C}-f_{22}M(k+\varepsilon)+f_{44}M(k-\varepsilon)=0,$$

again instead of three equations we get two ones.

Therefore we have derive a new system, let us present it in the matrix form

$$A_{14\times12}X_{12\times1}=0,\quad X=the\ column\{f_{11},f_{21},f_{41},f_{12},f_{22},f_{32},f_{23},f_{33},f_{43},f_{14},f_{34},f_{44}\}$$

A rank of the matrix A equals 12. If one removes the rows 1 and 2, the rink does not change. Determinant of the resulting matrix $A_{12\times12}$ is

$$\det A_{12\times12}=\frac{16iM\left(k^2-\varepsilon^2\right)^2\left(C^2+k^2-M^2-\varepsilon^2\right)\left(C^2-k^2-M^2+\varepsilon^2\right)^5}{C^2}.$$

Equating this determinant to zero, we find possible values for C:

$$C=\pm i\sqrt{\varepsilon^2-k^2-M^2},\quad multiplicity\ 5,$$
$$C=\pm\sqrt{\varepsilon^2-k^2+M^2},\quad multiplicity\ 2. \tag{26}$$

According to eq. (25), we have restriction $C^2<0$.

Therefore, the roots from the first line are appropriate.

For definiteness, we take

$$C=+i\sqrt{\varepsilon^2-k^2-M^2}\quad\Rightarrow\quad\varepsilon^2-k^2-M^2=-C^2. \tag{27}$$

Remembering on the quantization rules for C^2, we will use the formula

$$\varepsilon^2-k^2-M^2=2B(n_3+A_3+\frac{m}{2}+\frac{1}{4}); \tag{28}$$

where

$$A_3=\frac{1}{4}(-2m+1)>0,\quad m=0,-1,-2,...;\qquad A_3=\frac{1}{4}(2m+1)>0,\quad m=0,+1,+2,....$$

Now we are to study the above system at $C=+i\sqrt{\varepsilon^2-k^2-M^2}=iE$: its rank equals 7.

One may eliminate the row 1, 2, 4, 6, and 9.

Then remove on the right side the columns 1, 2, 4, 7, 10.
As a result, we obtain the nonhomogeneous system

$$A_{7\times7}X_{7\times1}=R_{7\times1},$$

where

$$A_{7\times7}=\begin{vmatrix} i(\varepsilon-k) & 0 & 0 & 0 & -iM & 0 & 0 \\ 0 & 0 & i(k+\varepsilon) & 0 & 0 & -iM & 0 \\ 0 & 0 & 0 & iE & i(\varepsilon-k) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i(k+\varepsilon) & iE \\ -\frac{iM(k-\varepsilon)}{E} & 0 & 0 & i(k+\varepsilon) & -\frac{i(k-\varepsilon)(k+\varepsilon)}{E} & 0 & 0 \\ 0 & i(k+\varepsilon) & \frac{iM(k+\varepsilon)}{E} & 0 & 0 & -\frac{i(k-\varepsilon)(k+\varepsilon)}{E} & i(\varepsilon-k) \\ 0 & -M(k+\varepsilon) & -\frac{(k+\varepsilon)\left(2k^2+M^2-2\varepsilon^2\right)}{E} & 0 & 0 & \frac{M\left(\varepsilon^2-k^2\right)}{E} & M(k-\varepsilon) \end{vmatrix},\; X_{7\times1}=\begin{vmatrix} f_{41} \\ f_{22} \\ f_{32} \\ f_{33} \\ f_{43} \\ f_{34} \\ f_{44} \end{vmatrix},$$

$$R_{7\times1}=\begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \\ i(k-\varepsilon) \\ 0 \\ 0 \end{vmatrix} f_{11}+\begin{vmatrix} iM \\ 0 \\ 0 \\ 0 \\ -\frac{i(k-\varepsilon)(k+\varepsilon)}{E} \\ 0 \\ 0 \end{vmatrix} f_{21}+\begin{vmatrix} 0 \\ iM \\ 0 \\ 0 \\ 0 \\ -\frac{i(k-\varepsilon)(k+\varepsilon)}{E} \\ -\frac{M(k-\varepsilon)(k+\varepsilon)}{E} \end{vmatrix} f_{12}+\begin{vmatrix} -i(k+\varepsilon) \\ 0 \\ iM \\ 0 \\ \frac{iM(k+\varepsilon)}{E} \\ 0 \\ 0 \end{vmatrix} f_{23}+\begin{vmatrix} 0 \\ i(k-\varepsilon) \\ 0 \\ iM \\ 0 \\ -\frac{iM(k-\varepsilon)}{E} \\ \frac{(k-\varepsilon)\left(2k^2+M^2-2\varepsilon^2\right)}{E} \end{vmatrix} f_{14};$$

Let us write down independent solutions; we write down their 16-dimensional form

$$1,\quad f_{11}=1,\; f_{21}=0,\; f_{12}=0,\; f_{23}=0,\; f_{14}=0,\quad \begin{vmatrix} f_{41} \\ f_{22} \\ f_{32} \\ f_{33} \\ f_{43} \\ f_{34} \\ f_{44} \end{vmatrix}=\begin{vmatrix} -\frac{E}{M} \\ 0 \\ 0 \\ \frac{(k-\varepsilon)^2}{M^2} \\ \frac{(k-\varepsilon)E}{M^2} \\ 0 \\ 0 \end{vmatrix};$$

$$2, \quad f_{11}=0,\ f_{21}=1,\ f_{12}=0,\ f_{23}=0,\ f_{14}=0, \quad \begin{vmatrix} f_{41} \\ f_{22} \\ f_{32} \\ f_{33} \\ f_{43} \\ f_{34} \\ f_{44} \end{vmatrix} = \begin{vmatrix} \frac{k+\varepsilon}{M} \\ 0 \\ 0 \\ \frac{(k-\varepsilon)E}{M^2} \\ -\frac{k^2+M^2-\varepsilon^2}{M^2} \\ 0 \\ 0 \end{vmatrix};$$

$$3, \quad f_{11}=0,\ f_{21}=0,\ f_{12}=1,\ f_{23}=0,\ f_{14}=0, \quad \begin{vmatrix} f_{41} \\ f_{22} \\ f_{32} \\ f_{33} \\ f_{43} \\ f_{34} \\ f_{44} \end{vmatrix} = \begin{vmatrix} 0 \\ \frac{E}{k+\varepsilon} \\ \frac{M}{k+\varepsilon} \\ 0 \\ 0 \\ 0 \\ 0 \end{vmatrix};$$

$$4, \quad f_{11}=0,\ f_{21}=0,\ f_{12}=0,\ f_{23}=1,\ f_{14}=0, \quad \begin{vmatrix} f_{41} \\ f_{22} \\ f_{32} \\ f_{33} \\ f_{43} \\ f_{34} \\ f_{44} \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 0 \\ -\frac{E}{M} \\ \frac{k+\varepsilon}{M} \\ 0 \\ 0 \end{vmatrix};$$

$$5, \quad f_{11}=0,\ f_{21}=0,\ f_{12}=0,\ f_{23}=0,\ f_{14}=1, \quad \begin{vmatrix} f_{41} \\ f_{22} \\ f_{32} \\ f_{33} \\ f_{43} \\ f_{34} \\ f_{44} \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{\varepsilon-k}{M} \\ -\frac{E}{M} \end{vmatrix}.$$

Recall that

$$A = \frac{i\left(f_{23}k + f_{41}k + f_{21}M - f_{43}M + f_{23}\varepsilon - f_{41}\varepsilon \right)}{2C},$$

$$D = \frac{i\left(- f_{14}k - f_{32}k + f_{12}M - f_{34}M + f_{14}\varepsilon - f_{32}\varepsilon \right)}{2C};$$

so we can find expressions A and B for each of five solutions

$$\begin{gathered} A_1 = \frac{\varepsilon - k}{M}, \quad A_2 = -\frac{E}{M}, \quad A_3 = 0, A_4 = 0, A_5 = 0; \\ D_1 = 0, \quad D_2 = 0, \quad D_3 = 0, \quad D_4 = 0, \quad D_5 = 0. \end{gathered} \tag{29}$$

Finally, we obtain 16-dimensional presentations for five solutions (let $\sqrt{\varepsilon^2 - M^2 - k^2} = E$):

$$\varphi_1 = \begin{vmatrix} f_{11} \\ f_{21} \\ f_{31} \\ f_{41} \\ f_{12} \\ f_{22} \\ f_{32} \\ f_{43} \\ f_{13} \\ f_{23} \\ f_{33} \\ f_{43} \\ f_{14} \\ f_{24} \\ f_{34} \\ f_{44} \end{vmatrix} = \begin{vmatrix} F_1(r) \\ 0 \\ \frac{\varepsilon - k}{M} F_1(r) \\ -\frac{E}{M} F_3(r) \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{\varepsilon - k}{M} F_1(r) \\ 0 \\ \frac{(k-\varepsilon)^2}{M^2} F_1(r) \\ \frac{(k-\varepsilon)E}{M^2} F_3(r) \\ 0 \\ 0 \\ 0 \\ 0 \end{vmatrix}, \quad \varphi_2 = \begin{vmatrix} f_{11} \\ f_{21} \\ f_{31} \\ f_{41} \\ f_{12} \\ f_{22} \\ f_{32} \\ f_{43} \\ f_{13} \\ f_{23} \\ f_{33} \\ f_{43} \\ f_{14} \\ f_{24} \\ f_{34} \\ f_{44} \end{vmatrix} = \begin{vmatrix} 0 \\ F_3(r) \\ -\frac{E}{M} F_1(r) \\ \frac{k+\varepsilon}{M} F_3(r) \\ 0 \\ 0 \\ 0 \\ 0 \\ -\frac{E}{M} F_1(r) \\ 0 \\ \frac{(k-\varepsilon)E}{M^2} F_1(r) \\ \frac{E^2}{M^2} F_3(r) \\ 0 \\ 0 \\ 0 \\ 0 \end{vmatrix},$$

$$\varphi_3=\begin{vmatrix} f_{11}\\ f_{21}\\ f_{31}\\ f_{41}\\ f_{12}\\ f_{22}\\ f_{32}\\ f_{43}\\ f_{13}\\ f_{23}\\ f_{33}\\ f_{43}\\ f_{14}\\ f_{24}\\ f_{34}\\ f_{44} \end{vmatrix}=\begin{vmatrix} 0\\ 0\\ 0\\ 0\\ F_3(r)\\ \frac{E}{k+\varepsilon}F_2(r)\\ \frac{M}{k+\varepsilon}F_3(r)\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0 \end{vmatrix},\varphi_4=\begin{vmatrix} f_{11}\\ f_{21}\\ f_{31}\\ f_{41}\\ f_{12}\\ f_{22}\\ f_{32}\\ f_{43}\\ f_{13}\\ f_{23}\\ f_{33}\\ f_{43}\\ f_{14}\\ f_{24}\\ f_{34}\\ f_{44} \end{vmatrix}=\begin{vmatrix} 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ F_3(r)\\ -\frac{E}{M}F_1(r)\\ \frac{k+\varepsilon}{M}F_3(r)\\ 0\\ 0\\ 0\\ 0 \end{vmatrix},\varphi_5=\begin{vmatrix} f_{11}\\ f_{21}\\ f_{31}\\ f_{41}\\ f_{12}\\ f_{22}\\ f_{32}\\ f_{43}\\ f_{13}\\ f_{23}\\ f_{33}\\ f_{43}\\ f_{14}\\ f_{24}\\ f_{34}\\ f_{44} \end{vmatrix}=\begin{vmatrix} 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ F_3(r)\\ 0\\ \frac{\varepsilon-k}{M}F_3(r)\\ -\frac{E}{M}F_2(r) \end{vmatrix}.$$

Conclusions

16-component system of equations describing the Dirac – Kähler particle in the presence of the external magnetic field has been solved. On the searched solutions we diagonalize operators of the energy, third projection of the total angular momentum, and the third projection of the linear momentum.

After separating the variables, we derive the system of sixteen first order differential equations in polar coordinate. To resolve this system, we apply the method by Fedorov – Gronskiy based on projective operator constructed from generator J^{12} for the field under consideration. According to this approach, we decompose the complete wave function into three 16-dimensional projective constituents, each expressed through only one functions of the polar coordinate. In the present system, these three basic functions are constructed in terms of the confluent hypergeometric functions, at this a quantization rule arises from the polynomial requirement. The 16-dimensional structure of solutions is determined by the arising linear algebraic system of equations. In this way, we find five independent solutions for the Dirac–Kähler particle in the magnetic field.

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