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RELATIVISTIC WAVE EQUATIONS WITH EXTENDED SET OF THE LORENTZ GROUP PRESENTATIONS

It is shown that the use of extended sets of irreducible representations of the Lorentz group opens new possibilities for the theory of relativistic wave equations from the point of view of the space-time description of both the internal structure and the isospin degrees of freedom of elementary particles. The approach developed in this work also makes it possible to apply the methods of the theory of relativistic wave equations in superstring and gauge models of fundamental interactions.

Key words: *relativistic wave equations, quantization, spin, statistics.*

Релятивистские волновые уравнения с расширенным набором представлений группы Лоренца

Показано, что использование расширенных наборов неприводимых представленний группы Лоренца открывает новые возможности теории релятивистских волновых уравнений с точки зрения пространственно-временного описания как внутренней структуры, так и изоспиновых степеней свободы элементарных частиц. Развиваемый в работе подход позволяет также применять методы теории релятивистских волновых уравнений в суперструнных и калибровочных моделях фундаментальных взаимодействий.

Ключевые слова: *релятивистские волновые уравнения, квантование, спин, статистика.*

Introduction

The Dirac equation served as the starting model for the creation of the general theory of relativistic wave equations (RWE) – first-order relativistic quantum-mechanical equations written in matrix-differential form.

The fundamental idea of this theory is the governing of any RWE with a corresponding set of irreducible representations of the group of geometric (space-time) symmetries of the Minkowski space.

We can formulate the following postulate basis for this theory [1; 2]:

1) any RWE must satisfy the invariance requirements with respect to the transformations of the proper Lorentz group and operation of the spatial reflection, also possibility of the Lagrangian formulation of the theory is assumed;

2) RWE describing a single physical micro-object should not be decayed in the sense of the full Lorentz group;

3) among the states of the micro-object, there cannot be those which correspond to zero energy;

4) the correct RWE must lead to a positive definite density of energy (charge) in the case of a whole (half-integer) spin;

5) fields with integral (and half-integral) spins are described on the basis of tensor (spinor) representations of the Lorentz group;

6) usually when constructing for a particle with spin a corresponding RWE we restrict ourselves to the minimally necessary set of irreducible representations of the Lorentz group.

The listed provisions of the theory of RWE were formulated in the 20s – 50s of the last century. They were based on the idea that elementary particles are nonstructural point-like micro objects with a single internal degree of freedom (spin), the latter has a spatio-temporal interpretation.

However, with the establishment of new experimental facts (the existence of internal structure for some particles, the presence of additional internal degrees of freedom besides spin, etc.), the above ideas have undergone significant changes. The very concept of «elementary particle» has also changed.

There arose the idea of existence of fundamentally new physical objects that unify the qualities of micro-particles (fields) with nonzero and zero mass (for example, the electroweak fields) and the properties of massless micro-objects with different helicity values (fields interacting with non-closed strings).

The Dirac – Kähler [3] and Petrash [4] equations were the first successful attempts to go beyond the postulates 1) – 6) and thereby greatly expanded the capabilities of the theory of RWE. They showed that if the postulate 6) is abandoned, the possibility of a spatio-temporal description of both the internal structure and the isospin degrees of freedom of particles appears. Since the middle of the 1960s, this direction began to develop actively in a number of scientific centers of the Republic of Belarus on the initiative and under the leadership of Academician of the Academy of Sciences of Belarus F. I. Fedorov.

Over the past decade, a wealth of results in the theory of RWE with an extended set of representations of the Lorentz group had been accumulated. In this paper, we present some significant results that can be adapted to modern experimental achievements and theoretical trends in high-energy physics.

1. RWEs with an extended set of the Lorentz group representations and internal structure of microobjects

A characteristic feature of all RWE considered above consists in the fact that they are based on the sets of the Lorentz group irreducible representations which are minimally necesary for constructing theory for a given spin. Along with that, in accordance with the ideology of the relativistic quantum mechanics, which interprets elementary particles as point-like structureless objects, such RWEs take into account only spin properties of particles. A possibility of describing other internal properties of particles in the orthodox version of the RWE theory is not provided.

Relaxing the requirement of minimality in usage of sets of the Lorentz group irreducible representations opens new possibilities of the RWE theory approach for a spatio-temporal (geometrized) description of internal properties of particles. To obtain equations which do not disintegrate with respect to the Lorentz group and which are capable of reflecting an internal structure of a particle with spin s, one can use the following possibilities: Either to include into a linking scheme representations with higher weights or to employ multiple representations of the Lorentz group. In the present chapter we show how to describe an internal electromagnetic structure of particles with lowest spins in the framework of the RWE theory with extended sets of the Lorentz group representations.

For the first time, the RWE for a particle with spin $s = 1/2$, which arises after involving additional – with respect to the bispinor – irreducible components in the representation space of a wavefunction, was proposed by Petras and Ulegla [5].

Here we give a brief description of the theory of the Petras equation in the Gel’fand – Yaglom approach. To this end, we consider the following linking scheme

$$
\begin{array}{ccc}
(0,\frac{1}{2})' & - & (\frac{1}{2},0)' \\
| & & | \\
(\frac{1}{2},1) & - & (1,\frac{1}{2}) \\
| & & | \\
(0,\frac{1}{2}) & - & (\frac{1}{2},0)
\end{array}
\qquad (1.1)
$$

Let us enumerate the irreducible representations contained in (1.1):

$$\left(0,\frac{1}{2}\right)\sim 1,\quad (0,\frac{1}{2})'\sim 2,\quad \left(1,\frac{1}{2}\right)\sim 3,$$

$$(\frac{1}{2},0)\sim 4,\quad (\frac{1}{2},0)'\sim 5,\quad (\frac{1}{2},1)\sim 6, \qquad (1.2)$$

Then, we for the spin blocks $C^{1/2}$, $C^{3/2}$ of the matrix

$$\Gamma_4 = (C^{1/2}\otimes I_2)\oplus(C^{3/2}\otimes I_4) \qquad (1.3)$$

we obtain the following expressions

$$C^{1/2} = \begin{pmatrix} 0 & 0 & 0 & c_{14}^{1/2} & 0 & c_{16}^{1/2} \\ 0 & 0 & 0 & 0 & c_{25}^{1/2} & c_{26}^{1/2} \\ 0 & 0 & 0 & c_{34}^{1/2} & c_{35}^{1/2} & c_{36}^{1/2} \\ c_{41}^{1/2} & 0 & c_{43}^{1/2} & 0 & 0 & 0 \\ 0 & c_{52}^{1/2} & c_{53}^{1/2} & 0 & 0 & 0 \\ c_{61}^{1/2} & c_{62}^{1/2} & c_{63}^{1/2} & 0 & 0 & 0 \end{pmatrix},\quad C^{3/2} = \begin{pmatrix} 0 & c_{36}^{3/2} \\ c_{63}^{3/2} & 0 \end{pmatrix}. \qquad (1.4)$$

To exclude spin $s = 3/2$, we impose constraints

$$c_{36}^{3/2} = c_{63}^{3/2} = 0 \text{ or } C^{3/2} = 0, \qquad (1.5)$$

from whence it follows

$$c_{36}^{1/2} = c_{63}^{1/2} = 0. \qquad (1.6)$$

The condition of the P-invariance leads to the relations

$$c_{14}^{1/2} = c_{14}^{1/2}, c_{25}^{1/2} = c_{52}^{1/2}, c_{16}^{1/2} = c_{43}^{1/2},$$

$$c_{26}^{1/2} = c_{53}^{1/2}, c_{34}^{1/2} = c_{61}^{1/2}, c_{35}^{1/2} = c_{62}^{1/2}. \qquad (1.7)$$

A possibility of the Lagrangian formulation implies

$$c_{14}^{1/2}, c_{25}^{1/2} \in R;\ c_{34}^{1/2} = \frac{\eta_{63}^{1/2}}{\eta_{14}^{1/2}}(c_{16}^{1/2})^*, c_{35}^{1/2} = \frac{\eta_{63}^{1/2}}{\eta_{25}^{1/2}}(c_{26}^{1/2})^*. \qquad (1.8)$$

With account of the constraints (1.5) – (1.8) the spin block $C^{1/2}$ acquires the form

$$C^{1/2} = \begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix},\ C = \begin{pmatrix} c_1 & 0 & c_3 \\ 0 & c_2 & c_4 \\ f_1 c_3^* & f_2 c_4^* & 0 \end{pmatrix}, \qquad (1.9)$$

where the notations

$$c_1 = c_{14}^{1/2}, \qquad c_2 = c_{25}^{1/2}, \qquad c_3 = c_{16}^{1/2}, \qquad c_4 = c_{26}^{1/2},$$

$$f_1 = \frac{\eta_{63}^{1/2}}{\eta_{14}^{1/2}}\,,\; f_2 = \frac{\eta_{63}^{1/2}}{\eta_{25}^{1/2}} \qquad (1.10)$$

are introduced for brevity.

A characteristic equation for the block *C* reads

$$\lambda^3 - (c_1 + c_2)\lambda^2 + (c_1 c_2 - f_1|c_3|^2 - f_2|c_4|^2)\lambda + f_1 c_2|c_3|^2 + f_2 c_1|c_4|^2 = \qquad (1.11)$$

To obtain a single mass value, it is necessary to impose

$$c_1 c_2 - f_1|c_3|^2 - f_2|c_4|^2 = 0,$$
$$f_1 c_2|c_3|^2 + f_2 c_1|c_4|^2 = 0. \qquad (1.12)$$

Without loss of generality, we choose the only nonzero eigenvalue of the block *C* to be

$$\lambda = c_1 + c_2 = 1. \qquad (1.13)$$

Such a choice yields the following minimal polynomials for the $C^{1/2}$ and the matrix Γ_4:

$$(C^{1/2})^2\left[(C^{\frac{1}{2}})^2 - 1\right] = 0,\; \Gamma_4^2(\Gamma_4^2 - 1) = 0. \qquad (1.14)$$

It remains to impose the condition of the charge definiteness, which in the present case ($C^{3/2}$, *n* = 2) acquires the form

$$\mathrm{Sp}\,((C^{1/2})^3\eta^{1/2}) \neq 0, \qquad (1.15)$$

where

$$\eta^{1/2} = \begin{pmatrix} 0 & \eta^{\square} \\ \eta^{\square} & 0 \end{pmatrix},\; \eta^{\square} = \begin{pmatrix} \eta_{14}^{1/2} & 0 & 0 \\ 0 & \eta_{25}^{1/2} & 0 \\ 0 & 0 & \eta_{36}^{1/2} \end{pmatrix}. \qquad (1.16)$$

It follows with account of (1.15) that

$$\eta_{14}^{1/2}c_1^3 + \eta_{25}^{1/2}c_2^3 + (2\eta_{63}^{1/2} + \eta_{14}^{1/2})c_1|c_3|^2 + (2\eta_{63}^{1/2} + \eta_{25}^{1/2})c_2|c_4|^2 \neq 0. \qquad (1.17)$$

A simultaneous fulfillment of the conditions (1.12), (1.13), (1.17) is ensured, e. g., by the choice

$$c_1 = \frac{1}{3},\; c_2 = \frac{2}{3},\; c_3 = \frac{\sqrt{2}}{3},\; c_4 = \frac{2}{3}, \qquad (1.18)$$

$$\eta_{14}^{1/2} = -1,\; \eta_{25}^{1/2} = 1,\; \eta_{36}^{1/2} = 1, \qquad (1.19)$$

giving the following matrices *C* and $\eta^{\square}$

$$C = \begin{pmatrix} 1/3 & 0 & \sqrt{2}/3 \\ 0 & 2/3 & 2/3 \\ -\sqrt{2}/3 & 2/3 & 0 \end{pmatrix}, \quad \eta^{\square} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \qquad (1.20)$$

Thus, we obtained the 20-component RWE with the linking scheme (1.1) featuring the nondiagonalizable matrix Γ_4.

This equation describes a spin-1/2 particle and obeys all physical requirements.

Let us now show that on the basis of the linking scheme (3.1) it is also possible to construct a RWE for a particle with spin *s* = 3/2 [6].

Sticking with the previous labelling of the irreducible representations contained in (1.1), we again arrive at the general form (1.4) of the spin blocks $C^{1/2}$, $C^{3/2}$ of the matrix Γ_4 in the Gel'fand – Yaglom basis.

The condition of the relativistic invariance leads to the following constraints

$$c_{36}^{3/2} = 2c_{36}^{1/2},\ c_{63}^{3/2} = 2c_{63}^{1/2}. \tag{1.21}$$

The RWE invariance with respect to spatial reflections gives besides (1.7) the relations

$$c_{36}^{1/2} = c_{63}^{1/2},\ c_{36}^{3/2} = c_{63}^{3/2}. \tag{1.22}$$

A possibility of the Lagrangian formulation of the theory completes the relations (1.7) by the condition

$$c_{36}^{1/2} \in R. \tag{1.23}$$

Since in the present case we are interested in spin 3/2, we can set without loss of generality

$$c_{36}^{3/2} = c_{63}^{3/2} = 1. \tag{1.24}$$

Under this choice the conditions (1.21) – (1.23) yield the following spin blocks $C^{1/2}$ and $C^{3/2}$:

$$C^{1/2} = \begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix},\ C = \begin{pmatrix} c_1 & 0 & c_3 \\ 0 & c_2 & c_4 \\ f_1 c_3^* & f_2 c_4^* & 0 \end{pmatrix}, \tag{1.25}$$

$$C^{3/2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{1.26}$$

where the notations (1.10) are used.

A characteristic equation for the spin block *C* has the form

$$\lambda^3 - \left(c_1 + c_2 + \tfrac{1}{2}\right)\lambda^2 + \left(\tfrac{c_1 c_2}{2} + c_1 c_2 - f_1|c_3|^2 - f_2|c_4|^2\right)\lambda$$
$$-\tfrac{c_1 c_2}{2} + f_1 c_2|c_3|^2 + f_2 c_1|c_4|^2 = 0. \tag{1.27}$$

To exclude states with spin 1/2 we must claim that all eigenvalues of the block $C^{1/2}$ are equal to zero. This requirement leads to the conditions

$$c_1 + c_2 + \tfrac{1}{2} = 0,$$
$$\tfrac{c_1 c_2}{2} + c_1 c_2 - f_1|c_3|^2 - f_2|c_4|^2 = 0, \tag{1.28}$$
$$-\tfrac{c_1 c_2}{2} + f_1 c_2|c_3|^2 + f_2 c_1|c_4|^2 = 0,$$

where the numbers f_1 and f_2 may independently of each other take values either +1 or −1.

$$(C^{1/2})^3 = 0,\ (C^{3/2})^2 - 1 = 0, \tag{1.29}$$

$$\Gamma_4^3(\Gamma_4^2 - 1) = 0. \tag{1.30}$$

For example, let us choose

$$f_1 = -1\,, \quad f_2 = -1, \tag{1.31}$$

$$c_1 = \frac{1}{2},\ c_2 = -1,\ c_3 = \frac{1}{2\sqrt{3}},\ |c_4| = \sqrt{\frac{2}{3}}, \tag{1.32}$$

realizing one of the admissible possibilities.

This choice leads to the matrix of the bilinear form having blocks

$$\eta^{1/2} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}, \quad \eta^{3/2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{1.33}$$

as well as to the spin block $C^{1/2}$ of the matrix Γ_4

$$C^{1/2} = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2\sqrt{3}} \\ 0 & 0 & 0 & 0 & 1 & \sqrt{\frac{2}{3}} \\ 0 & 0 & 0 & -\frac{1}{2\sqrt{3}} & -\sqrt{\frac{2}{3}} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2\sqrt{3}} & 0 & 0 & 0 \\ 0 & -1 & \sqrt{\frac{2}{3}} & 0 & 0 & 0 \\ -\frac{1}{2\sqrt{3}} & -\sqrt{\frac{2}{3}} & \frac{1}{2} & 0 & 0 & 0 \end{pmatrix}. \tag{1.34}$$

By virtue of (1.30) the charge definiteness condition for $n = 3$ acquires the form

$$(\mathrm{Sp}\,(\Gamma_4^4\eta))^2 - \left(\mathrm{Sp}\,(\Gamma_4^3\eta)\right)^2 < 0. \tag{1.35}$$

Using the explicit form of the matrices Γ_4 and η, we obtain the relations

$$\mathrm{Sp}\,(\Gamma_4^4\eta) = 0, \quad \mathrm{Sp}\,(\Gamma_4^3\eta) = 8, \tag{1.36}$$

which ensure a fulfillment of the inequality (1.35).

To construct RWEs describing microobjects with spins $s = 0$ and $s = 1$ and differing from the well-known Duffin – Kemmer equations, we consider the following set of the Lorentz group irreducible representations [7; 8]:

$$(0,0)\oplus 2\left(\frac{1}{2},\frac{1}{2}\right)\oplus(0,1)\oplus(1,0), \tag{1.37}$$

which constitute the linking scheme

$$\begin{array}{c} (0,0) \\ | \\ (0,1) - 2\left(\frac{1}{2},\frac{1}{2}\right) - (1,0). \end{array} \tag{1.38}$$

Here the vector representation ($\frac{1}{2}$; $\frac{1}{2}$) has the multiplicity two. In the following, we label one of them with the prime to distinguish the two vector representations from each other.

The block structure of the matrix Γ_4 corresponding to the RWE based on the scheme (1.37) has the form

$$\Gamma_4 = \begin{pmatrix} C^0 & 0 \\ 0 & C^1 \otimes I_3 \end{pmatrix} \tag{1.39}$$

in the Gel'fand – Yaglom basis. Labeling the irreducible components contained in (1.37) by

$$(0,0)\sim 1, \quad (\frac{1}{2},\frac{1}{2})'\sim 2, \quad \left(\frac{1}{2},\frac{1}{2}\right)\sim 3, \quad (0,1)\sim 4, \quad (1,0)\sim 5,$$

we obtained after applying the conditions of the relativistic and *P*-invariance conditions the following spin blocks

$$C^0 = \begin{pmatrix} 0 & c_{12}^0 & c_{13}^0 \\ c_{21}^0 & 0 & 0 \\ c_{31}^0 & 0 & 0 \end{pmatrix}, \; C^1 = \begin{pmatrix} 0 & 0 & c_{24}^1 & c_{24}^1 \\ 0 & 0 & c_{34}^1 & c_{34}^1 \\ c_{43}^1 & c_{43}^1 & 0 & 0 \\ c_{42}^1 & c_{43}^1 & 0 & 0 \end{pmatrix}. \tag{1.40}$$

The matrix of the bilinear form in the Gel'fand – Yaglom basis reads

$$\eta = \begin{pmatrix} \eta^0 & 0 \\ 0 & \eta^0 \otimes I_3 \end{pmatrix},$$

$$\eta^0 = \begin{pmatrix} \eta_{11}^0 & 0 & 0 \\ 0 & \eta_{22}^0 & 0 \\ 0 & 0 & \eta_{33}^0 \end{pmatrix}, \tag{1.41}$$

$$\eta^1 = \begin{pmatrix} \eta_{22}^1 & 0 & 0 & 0 \\ 0 & \eta_{33}^1 & 0 & 0 \\ 0 & 0 & 0 & \eta_{45}^1 \\ 0 & 0 & \eta_{54}^1 & 0 \end{pmatrix}.$$

Choosing its nonzero elements to be

$$\eta_{11}^0 = -\eta_{22}^0 = \eta_{33}^0 = \eta_{22}^1 = -\eta_{33}^1 = -\eta_{45}^1 = -\eta_{54}^1 = 1, \tag{1.42}$$

we reduce the condition (1.48) of a possibility of the Lagrangian formulation to the relations

$$c_{21}^0 = -(c_{12}^0)^*, \; c_{31}^0 = (c_{13}^0)^*, \; c_{42}^1 = -(c_{24}^1),^* \; c_{43}^1 = (c_{34}^1)^*. \tag{1.43}$$

The remaining arbitrariness in choice of values for the matrix Γ_4 elements we can use for obtaining RWE with a desired value of spin.

Thus, setting

$$c_{12}^0 = 0, \; c_{13}^0 = c_{24}^1 = c_{34}^1 = 1, \tag{1.44}$$

we obtain the spin blocks

$$C^0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \; C^1 = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{pmatrix}. \tag{1.45}$$

It is easy to check the spin blocks C^1 and C^0 and the matrix Γ_4 obey the following minimal equations

$$(C^1)^3 = 0, \; C^0[(C^0)^2 - 1] = 0, \tag{1.46}$$

$$\Gamma_4^3(\Gamma_4^2 - 1) = 0. \tag{1.47}$$

From (1.46) it follows that to the state with spin *s* = 0 corresponds the only mass value, while all eigenvalues of the block C^1 are zero, i.e. states with *s* = 1 are absent.

Thus, we obtain the RWE for a microobject with spin $s = 0$ and a single mass value.

Using the relations (1.41), (1.42), and (1.45) for matrices Γ_4 and η it is easy to check the identities

$$\mathrm{Sp}\,(\Gamma_4^3\eta) = 0,\ \ \mathrm{Sp}\,(\Gamma_4^4\eta) = 2. \tag{1.48}$$

They ensure the fulfillment of the energy definiteness condition which for the present case ($n = 3$) is expressed by the inequality

$$(-1\,)^4[(\mathrm{Sp}\,(\Gamma_4^4\eta))^2 - (\mathrm{Sp}\,(\Gamma_4^3\eta))^2] > 0. \tag{1.49}$$

In the tensor formulation the constructed RWE reads

$$\begin{gathered}\partial_\mu\psi_\mu + m\psi_0 = 0,\\ \partial_\nu\psi_{[\mu\nu]} + \partial_\mu\psi_0 + m\psi_\mu = 0,\\ -\partial_\nu\psi_{[\mu\nu]} + m\psi_\mu^{\square} = 0,\\ -\partial_\mu\psi_\nu + \partial_\nu\psi_\mu - \partial_\mu\psi_\nu^{\square} + \partial_\nu\psi_\mu^{\square} + m\psi_{[\mu\nu]} = 0,\end{gathered} \tag{1.50}$$

where ψ_0 is a scalar, ψ_μ and $\psi_\mu^{\square}$ are 4-vectors, $\psi_{[\mu\nu]}$ is an antisymmetric tensor of the second rank. From (1.50) it is easy to derive the second-order equation

$$(\square - m^2)\psi_0 = 0, \tag{1.51}$$

which means that the system (1.50) indeed describes a particle with nonzero mass and spin $s = 0$.

To construct a RWE for a microparticle with spin $s = 1$ on the basis of the linking scheme (1.38), we choose the following values for the matrix elements of the matrix (1.41):

$$\eta_{11}^0 = \eta_{22}^0 = -\eta_{33}^0 = -\eta_{22}^1 = \eta_{33}^1 = \eta_{45}^1 = \eta_{54}^1 = 1. \tag{1.52}$$

Then, according to the condition (1.48) we obtain

$$c_{21}^0 = (c_{12}^0)^*,\ c_{31}^0 = -(c_{13}^0)^*,\ c_{42}^1 = -(c_{24}^1),^*\ c_{43}^1 = (c_{34}^1)^* \tag{1.53}$$

Within the remaining freedom in choice of the matrix elements of the spin blocks C^0 and C^1 (1.40) we make a particular selection of values

$$\begin{gathered}c_{12}^0 = c_{13}^0 = c_{21}^0 = -c_{31}^1 = 1,\\ c_{24}^1 = c_{42}^1 = 0, \qquad c_{34}^1 = c_{43}^1 = \frac{1}{\sqrt{2}}.\end{gathered} \tag{1.54}$$

Thus, we arrive at the final form of the matrices η^0, η^1, C^0, and C^1

$$\eta^0 = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix},\ \eta^1 = \begin{pmatrix}-1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{pmatrix}. \tag{1.55}$$

$$C^0 = \begin{pmatrix}0 & 1 & 1\\ 1 & 0 & 0\\ -1 & 0 & 0\end{pmatrix},\ C^1 = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 1 & \pm1\\ 0 & 1 & 0 & 0\\ 0 & \pm1 & 0 & 0\end{pmatrix}. \tag{1.56}$$

It is easy to check that the minimal equations for the spin blocks (3.56) of the matrix Γ_4 have the form

$$(C^0)^3 = 0,\ C^1[(C^1)^2 - 1] = 0. \tag{1.57}$$

This means that the corresponding RWE indeed describes a microparticle with spin *s* = 1.

From (1.57) it follows that the minimal equation for the matrix Γ_4 coincides with its analog (1.47) for a scalar particle. Therefore the condition of the energy definiteness in the present case should coincide with (1.49). Using the definitions (1.55) and (1.56), one can verify that the condition (1.49) is valid for a vector particle as well.

A tensor formulation of the obtained RWE with the extended set of representations for a particle with spin *s* = 1 reads

$$\begin{gathered}\partial_{\mu}\psi_{\mu} + \partial_{\mu}\psi_{\mu}^{?} + m\psi_0 = 0,\\ \partial_{\nu}\psi_{[\mu\nu]} + \partial_{\mu}\psi_0 + m\psi_{\mu} = 0,\\ \partial_{\mu}\psi_0 + m\psi_{\mu}^{?} = 0,\\ -\partial_{\mu}\psi_{\nu} + \partial_{\nu}\psi_{\mu}^{?} + m\psi_{[\mu\nu]} = 0.\end{gathered} \tag{1.58}$$

From this system one can derive the equations

$$(\square - m^2)\left(\psi_{\mu} + \psi_{\mu}^{?}\right) = 0,\ \partial_{\mu}\left(\psi_{\mu} + \psi_{\mu}^{?}\right) = 0, \tag{1.59}$$

which unambiguously indicate that the system (1.58) does describe a vector particle with nonzero mass.

Other versions of the extended RWEs for particles with lowest spins are proposed in the papers [9] (spin 1/2), [10] (spin 0), and [11] (spin 1).

A question of the physical inequivalence of RWEs with minimal and extended sets of representations of the Lorentz group has been discussed for the first time for specific equations in the papers [7; 8] (spins 0 and 1), [12; 13] (spin 1/2), and [14] (spin 3/2). The essence and the main results of the latter study are the following.

First, one introduces minimal and extended equations for free particles

$$\left(\Gamma_{\mu}^{(0)}\partial_{\mu} + m\right)\Psi_0(x) = 0, \tag{1.60}$$

$$\left(\Gamma_{\mu}^{(1)}\partial_{\mu} + m\right)\Psi_1(x) = 0, \tag{1.61}$$

which are defined in the representation spaces corresponding to the irreducible Lorentz group representations T_0 and $\mathrm{T}_1 = \mathrm{T}_0 + T^{?}$, respectively.

Second, one finds an explicit form of operators *R* and *K* transforming Ψ_0 into Ψ_1 and vice versa:

$$R = (A, 0),\ K = \begin{pmatrix} F \\ G \end{pmatrix}, \tag{1.62}$$

$$R\Psi_1 = (A, 0)\begin{pmatrix} \Psi_1^0 \\ \Psi_1^1 \end{pmatrix} = A\Psi_1^0 = \Psi_0, \tag{1.63}$$

$$K\Psi_0 = \begin{pmatrix} F \\ G \end{pmatrix}\Psi_0 = \begin{pmatrix} F\Psi_0 \\ G\Psi_0 \end{pmatrix} = \begin{pmatrix} \Psi_1^0 \\ \Psi_1^1 \end{pmatrix} = \Psi_1. \tag{1.64}$$

Here *A* and *F* are rectangular number-valued matrices, satisfying the condition

$$AF = I, \tag{1.65}$$

while the matrix *G* in general contains differentiation operators. Moreover, the operators *R* and *K* should obey the relation

$$R\Gamma_{\mu}^{(1)}K = \Gamma_{\mu}^{(0)} + \mathrm{B}_{\mu}\,, \tag{1.66}$$

where the matrices B_μ satisfy the equation

$$\mathrm{B}_\mu \, \partial_\mu \Psi_0(x) = 0. \tag{1.67}$$

Next, one considers the equations

$$\left(\Gamma_\mu^{(0)} D_\mu + m\right)\Phi_0(x) = 0, \tag{1.68}$$

$$\left(\Gamma_\mu^{(1)} D_\mu + m\right)\Phi_1(x) = 0, \tag{1.69}$$

describing particles which interact with the electromagnetic field added by means of the minimal coupling

$$\partial_\mu \to D_\mu = \partial_\mu - ie\mathrm{A}_\mu. \tag{1.70}$$

The operator R, transforming $\Phi_1(x)$ into $\Phi_0(x)$ has the same form as for free particles. For the operator K' which realizes the inverse transformation we obtain

$$K' = \begin{pmatrix} F \\ G + G' \end{pmatrix}, \tag{1.71}$$

where the addition G' is caused by the derivative extension (1.70). In light of this, the equation (1.69) can be cast to the form

$$\left(R\Gamma_\mu^{(1)} D_\mu K' + m\right)\Phi_0(x) = 0, \tag{1.72}$$

or

$$\left(\Gamma_\mu^{(0)} D_\mu + Q + m\right)\Phi_0(x) = 0, \tag{1.73}$$

where

$$Q \sim R\Gamma_\mu^{(1)} D_\mu G' + B_\mu D_\mu. \tag{1.74}$$

Thus, after bringing the equation (1.69) into shape of the equation (1.68) for a wave-function with a minimally necessary number of components, we observe in the latter an occurrence of the additional term Q which is given by (1.74). For particles with spins $s = 1/2$ and $s = 3/2$ described by the RWEs with the extended sets of representations, which have been discussed above, this term acquires the form

$$Q \sim \frac{ie}{m}\left(\partial_\mu \mathrm{A}_\nu - \partial_\nu \mathrm{A}_\mu\right) J^{[\mu\nu]} = \frac{ie}{m} F_{[\mu\nu]} J^{[\mu\nu]}, \tag{1.75}$$

where $J^{[\mu\nu]}$ are the generators of the Lorentz group representations in the respective spaces.

In the nonrelativistic approximation, this term describes an additional – anomalous – magnetic moment, and it leads to an interaction of the Pauli type in the Lagrangian.

In the cases of the extended RWEs for particles with lowest spins, which have been discussed above, this additional term has the form

$$Q \sim \frac{e^2}{m} F_{[\mu\nu]} F_{[\mu\nu]} e^{00}, \tag{1.76}$$

$$Q \sim \frac{e^2}{m} F_{[\mu\nu]} F_{[\rho\sigma]} e^{[\mu\nu],[\rho\sigma]}, \tag{1.77}$$

where e^{AB} are the generalized Kronecker symbols defined by the formulas\

$$(e^{AB})CD = \delta_{AC}\delta_{BD}. \tag{1.78}$$

In the case of a particle with spin $s = 0$ the term Q given by (1.76) describes in the nonrelativistic approximation dipole electric and magnetic polarizablities of this particle which are induced by an external electromagnetic field. In the case of a particle with spin s = 1 the analogous term (1.77) describes a particle's static tensor electric polarizability.

It is obvious that an additional interaction with an external electromagnetic field should influence a form of matrix elements for specific scattering processes. Detailed calculations of some of these processes have been performed in the papers [7–14]. It has been shown that in the first order perturbation theory the mentioned interaction does not show up. For examples, scattering on the Coulomb centre happens in the same way for both types of RWEs based on the minimal and the extended sets of the Lorentz group representations. In turn, a calculation of cross-sections of the typical second order process – the Compton light scattering on particles described by the RWEs with extended sets of representations – leads in all cases to matrix elements having the form

$$M_1 = M_0 + M^{\square}, \tag{1.79}$$

where M_0 is a matrix element corresponding to a particle described by an RWE with a minimal set of representations, and $M^{\square}$ is a correction caused by the presence of an internal electromagnetic structure in this particle. Explicit expressions for these corrections can be found in the above mentioned publications.

Thus, a simple extension of a set of used representations, which can also be realized by an inclusion of replicated irreducible components, allows us to reflect internal particles' structure by means of the conventional spatiotemporal description in terms of the RWE theory. Apparently, in the conceptual respect this approach is more advantageous as compared to the popular phenomenological approach, in which additional terms describing specific structural effects are introduced into the Lagrangian by hand.

2. Tensor RWEs of the Dirac type and geometrized description of internal degrees of freedom of fundamental particles

Since an existence of additional internal quantum numbers for fundamental particles is by now a firmly established fact, there arises a question whether it is possible to apply the RWE theory for describing degrees of freedom associated with internal and, in particular, gauge symmetries. The traditional gauge theories of fundamental particles and their interactions are based as a rule on the Dirac equation whose wavefunction is equipped by a free non-Lorentzian index playing the role of an internal variable. From the RWE theory point of view, such an approach actually means a usage of equations which are falling apart under the Lorentz group transformations. On this basis are constructed the renowned models of electroweak and strong interactions, and the Standard $SU(3) \times SU(2) \times U(1)$ SU(3) Model.

However, this approach is not capable of solving a number of problems. In particular, it is not very effective in including the gravitational interaction into a general scheme. It is presently hoped that a solution of this problem lies in employing symmetry groups whose transformations would affect both spatiotemporal and internal variables on equal basis. In other words, it is a matter of an eventual geometrized introduction of internal degrees of freedom.

Let us briefly review the most known approaches in this direction:

• theories of the Kaluza – Klein type, which operate with the space-time of a dimension greater than four, additional dimensions being treated on equal footing with the four standard – observable – ones. A compactification of extra dimensions leads to a release of internal degrees of freedom retaining their geometrical character;

• supersymmetry-supergravitation, uniting particles with different spins and statistics into entire supermultiplets. One of the premises in this approach consists in the existence of a new mathematical structure – the supersymmetry transformations, which mix up together bosonic and fermionic fields.

In analogy with the Lorentz transformations which reveal the connection between the space and the time, the supersymmetry transformations unite the space-time and internal degrees of freedom of particles into the entire entity;

• string and superstring theories, which include the ideas of Kaluza and Klein, the supersymmetries, the gauge approach, and the general relativity.

It seems however feasible to formulate yet another approach of a geometrized description of internal degrees of freedom, which is based on the usage of an extended set of the Lorentz group representations (including multiple ones) in the framework of the RWE theory.

A natural possibility in this respect consists in using nondisintegrating – with respect to the full Lorentz group – equations whose wavefunction possesses transformation properties of a direct product of the Dirac bispinors, and whose matrices Γ_μ satisfy commutation relations of the Dirac matrices' algebra. In the following, we will call such RWEs Dirac-like RWEs, or RWEs of the Dirac type.

The most widely known RWE of the discussed type is the Dirac – Kähler (DK) equation, which represents itself the most general differential equation (or the system of equations) of the first order over the field of complex numbers for the full set of antisymmetric tensor fields in the Minkowski space.

On the other hand, in the appropriate basis (let us call it fermionic) a wavefunction of the DK equation possesses the Lorentz transformation properties of a direct product of the Dirac bispinor times the charge-conjugated bispinor. In the tensor formulation the DK equation can be represented by the system

$$\begin{gathered}
\partial_\mu \psi_\mu + m\psi_0 = 0, \\
\partial_\nu \psi_{[\mu\nu]} + \partial_\mu \psi_0 + m\psi_\mu = 0, \\
-\partial_\mu \psi_\nu + \partial_\nu \psi_\mu + i\varepsilon_{\mu\nu\alpha\beta}\partial_\alpha \widetilde{\psi_\beta} + m\psi_{[\mu\nu]} = 0, \\
\partial_\mu \psi_\mu + m\widetilde{\psi_0} = 0, \\
\partial_\nu \tilde{\psi}_{[\mu\nu]} + \partial_\mu \widetilde{\psi_0} + m\widetilde{\psi_\mu} = 0, \\
\frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}\partial_\nu \psi_{[\alpha\beta]} + \partial_\mu \widetilde{\psi_0} + m\widetilde{\psi_\mu} = 0.
\end{gathered} \tag{2.1}$$

Here ψ_0 is a scalar, ψ_μ is a vector, $\psi_{[\mu\nu]}$ is a second-rank antisymmetric tensor, $\widetilde{\psi_\mu} = \frac{1}{3!}\varepsilon_{\mu\nu\alpha\beta}\psi_{[\nu\alpha\beta]}$ is a pseudovector, which is dual conjugated to the third-rank antisymmetric tensor $\psi_{[\nu\alpha\beta]}$, and $\widetilde{\psi_0} = \frac{1}{4!}\varepsilon_{\mu\nu\alpha\beta}\psi_{[\mu\nu\alpha\beta]}$ is a pseudoscalar which is dual conjugated to the fourth-rank antisymmetric tensor $\psi_{[\mu\nu\alpha\beta]}$.

The system (2.1) is nondisintegrating in the sense of the full Lorentz group.

It can be written in the matrix-differential form, which is standard for the RWE theory, where the wavefunction Ψ is represented by the column vector with tensor-valued components

$$\Psi = (\psi_0, \widetilde{\psi_0}, \psi_\mu, \widetilde{\psi_\mu}, \psi_{[\mu\nu]})^T, \tag{2.2}$$

and the 16×16 matrices Γ_μ are expressed by

$$\begin{gathered}
\Gamma_\mu = \Gamma_\mu{}^{(+)} + \Gamma_\mu{}^{(-)}, \\
\Gamma_\mu{}^{(+)} = e^{\tilde{0}\tilde{\mu}} + e^{\tilde{\mu}\tilde{0}} + e^{\lambda,[\lambda\mu]} + e^{[\lambda\mu],\lambda},
\end{gathered} \tag{2.3}$$

$$\Gamma_\mu{}^{(-)} = e^{0\mu} + e^{\mu 0} + \frac{i}{2}\varepsilon_{\mu\nu\alpha\beta}\left(e^{\tilde{\lambda},[\alpha\beta]} + e^{\,[\alpha\beta],\tilde{\lambda}}\right).$$

Using the known rules of the generalized Kronecker symbols' multiplication, it is easy to check that the matrices Γ_μ (2.3) satisfy the Dirac matrices' algebra

$$\Gamma_\mu\Gamma_\nu + \Gamma_\nu\Gamma_\mu = 2\delta_{\mu\nu}. \tag{2.4}$$

For establishing the group of internal symmetry of the DK field one can conveniently pass from the tensor basis (2.2) to the fermionic one, in which the matrices Γ_μ and the matrix η of the Lorentz-invariant bilinear form read

$$\Gamma_\mu = I_4 \otimes \gamma_4, \tag{2.5}$$

$$\eta = \gamma_4 \otimes \gamma_4. \tag{2.6}$$

We remind that by an internal symmetry transformation of the RWE we understand a linear transformation of the wavefunction

$$\Psi'(x) = Q\Psi(x), \tag{2.7}$$

which does not touch spatiotemporal coordinates and which leaves invariant the equation and its Lagrangian. For this to happen, a matrix Q must satisfy the conditions

$$\left[\Gamma_\mu, Q\right]_- = 0, \tag{2.8}$$

$$Q^+\eta Q = \eta. \tag{2.9}$$

Applying the conditions (2.8), (2.9) to the matirces Γ_μ and η leads us to a noncompact 15-parametric group $SU(2,2)$, whose generators may serve the Hermitian matrices

$$\Gamma'_\mu,\ \Gamma'_5,\ i\Gamma'_\mu\Gamma'_5,\ i\Gamma'_{[\mu}\Gamma'_{\nu]} = \tfrac{i}{2}\left(\Gamma'_\mu\Gamma'_\nu - \Gamma'_\nu\Gamma'_\mu\right). \tag{2.10}$$

Here

$$\Gamma'_5 = \Gamma'_1\Gamma'_2\Gamma'_3\Gamma'_4 \tag{2.11}$$

and

$$\Gamma'_\mu = \Gamma_\mu{}^{(+)} - \Gamma_\mu{}^{(-)} \tag{2.12}$$

is the second set of 16×16 matrices, satisfying – like Γ_μ – the algebra of Dirac matrices and commuting with the matrices Γ_μ. In the fermionic basis these matrices have the form

$$\Gamma'_\mu = \gamma_4 \otimes I_4. \tag{2.13}$$

A characteristic feature of the internal symmetry group of the DK equation is that its generators (2.10) do not commute with the Lorentz generators

$$J_{[\mu\nu]} = \tfrac{1}{4}\left(\Gamma_{[\mu}\Gamma_{\nu]} + \Gamma'_{[\mu}\Gamma'_{\nu]}\right) \tag{2.14}$$

from the representation of the wavefunction Ψ. Along with this, the group G corresponding to the full invariance algebra of the DK equation appears to be a semidirect product of the Lorentz group Λ and the group of the internal symmetry Q: $G = \Lambda \oslash Q$. On the other hand, the group can be represented as a direct product $G = \Lambda' \otimes Q$, where Λ' is an overdefined Lorentz group with respect to which the wavefunction Ψ is no longer a collection of the tensor-valued components, but rather a collection of four Dirac fields with a usual internal symmetry (i. e. commuting with the Lorentz group transformations).

The above statements remain valid for all interactions, including gauge ones, which do not violate an internal symmetry of the free Lagrangian.

They assert that it is generally possible to apply the DK equation for a description of particles with spin *s* = 1/2 and internal degrees of freedom, which thus have a geometric origin (for details see, e. g., [15]).

The idea that the DK equation can be exploited as a geometric model for the generations of quarks (or leptons) was first put forward in the works [16; 17].

Let us now give a matrix formulation of the DK equation in the Gel'fand – Yaglom basis, which will be useful in the following.

We begin with the next set of irreducible representations of the proper Lorentz group

$$\begin{array}{c} 2(0,0) \\ | \\ (0,1) - 2\left(\frac{1}{2},\frac{1}{2}\right) - (1,0), \end{array} \tag{2.15}$$

containing twofold components $(0,0)$ and $\left(\frac{1}{2},\frac{1}{2}\right)$. The matrix Γ_4 of the corresponding RWE in the Gel'fand – Yaglom basis has the form $\Gamma_4 = \begin{pmatrix} C^0 & 0 \\ 0 & C^1 \otimes I_3 \end{pmatrix}$, where

$$C^0 = \begin{pmatrix} 0 & 0 & c_{13}^0 & c_{14}^0 \\ 0 & 0 & c_{23}^0 & c_{24}^0 \\ c_{31}^0 & c_{32}^0 & 0 & 0 \\ c_{41}^0 & c_{42}^0 & 0 & 0 \end{pmatrix}, C^1 = \begin{pmatrix} 0 & 0 & c_{35}^1 & c_{36}^1 \\ 0 & 0 & c_{45}^1 & c_{46}^1 \\ c_{53}^1 & c_{54}^1 & 0 & 0 \\ c_{63}^1 & c_{64}^1 & 0 & 0 \end{pmatrix}, \tag{2.16}$$

and the following labelling of the irreducible representations contained in (2.15) is adopted:

$$(0,0)\sim 1, \quad (0,0)'\sim 2, \quad \left(\frac{1}{2},\frac{1}{2}\right)\sim 3, \quad \left(\frac{1}{2},\frac{1}{2}\right)'\sim 4, \quad (0,1)\sim 5, \quad (1,0)\sim 6. \tag{2.17}$$

Here, like it was also the case earlier, the prime is used to distinguish between the multiple representations.

Let us first consider the spin block C^1.

The conditions of the relativistic and P-invariance of the theory impose on elements c_{ij}^1 in general case the constraints

$$c_{35}^1 = \pm c_{36}^1, \; c_{45}^1 = \pm c_{46}^1, \; c_{53}^1 = \pm c_{63}^1, \; c_{54}^1 = \pm c_{64}^1. \tag{2.18}$$

Here the choice of signs «+» or «–» depends on the definition of the spatial inversion operator. In the present context this means that the sign «+» («–») in (2.18) occurs for the true vectorial (pseudovectorial) character of the multiple representations $\left(\frac{1}{2},\frac{1}{2}\right)$.

It turns out that one can construct a RWE satisfying all necessary physical requirements, if one chooses one of the representations $\left(\frac{1}{2},\frac{1}{2}\right)$ as true vectorial and the other as pseudovectorial (which is denoted in the following as $\left(\frac{1}{2},\frac{1}{2}\right)'$). Then the relations (2.18) acquire the forms

$$c_{35}^1 = c_{36}^1, \; c_{45}^1 = -c_{46}^1, \; c_{53}^1 = c_{63}^1, \; c_{54}^1 = -c_{64}^1, \tag{2.19}$$

and for the block C^1 one gets an expression

$$C^1 = \begin{pmatrix} 0 & 0 & c_{35}^1 & c_{35}^1 \\ 0 & 0 & c_{45}^1 & -c_{45}^1 \\ c_{53}^1 & c_{54}^1 & 0 & 0 \\ c_{53}^1 & -c_{54}^1 & 0 & 0 \end{pmatrix}. \tag{2.20}$$

Analogously, one of the representations (0, 0) in (2.15) we choose as scalar and the other as pseudoscalar (also labelling it in the following with the prime).

And since in a P-invariant RWE a vector (pseudovector) representation cannot link with a pseudscalar (scalar) one, the following equalities should take place

$$c_{14}^0 = c_{23}^0 = c_{41}^0 = c_{33}^0 = 0. \tag{2.21}$$

With this in view, the block C^0 (2.18) is transformed to the form

$$C^0 = \begin{pmatrix} 0 & 0 & c_{13}^0 & 0 \\ 0 & 0 & 0 & c_{24}^0 \\ c_{31}^0 & 0 & 0 & 0 \\ 0 & c_{42}^0 & 0 & 0 \end{pmatrix}, \tag{2.22}$$

and the representations (2.15) build the linking scheme

$$\begin{array}{ccccc} & & (0,1) & & \\ & & \diagup\diagdown & & \\ (0,0)' - \left(\frac{1}{2},\frac{1}{2}\right)' & & & & \left(\frac{1}{2},\frac{1}{2}\right) - (0,0). \\ & & \diagdown\diagup & & \\ & & (1,0) & & \end{array} \tag{2.23}$$

The blocks η^0, η^1 of the matrix of the bilinear invariant form η have in this case the form

$$\eta = \eta^0 \oplus (\eta^1 \otimes I_3),$$

$$\eta^0 = \begin{pmatrix} \eta_{11}^0 & 0 & 0 & 0 \\ 0 & \eta_{22}^0 & 0 & 0 \\ 0 & 0 & \eta_{33}^0 & 0 \\ 0 & 0 & 0 & \eta_{44}^0 \end{pmatrix}, \ \eta^1 = \begin{pmatrix} \eta_{33}^1 & 0 & 0 & 0 \\ 0 & \eta_{44}^1 & 0 & 0 \\ 0 & 0 & 0 & \eta_{56}^1 \\ 0 & 0 & \eta_{65}^1 & 0 \end{pmatrix}. \tag{2.24}$$

$$\eta_{33}^1 = -\eta_{33}^0, \ \eta_{44}^1 = -\eta_{44}^0, \ \eta_{65}^1 = \pm\eta_{56}^1. \tag{2.25}$$

$$c_{31}^0 = \frac{\eta_{33}^0}{\eta_{11}^0}(c_{13}^0)^*, \ c_{42}^0 = \frac{\eta_{44}^0}{\eta_{22}^0}(c_{24}^0)^*, \ c_{53}^1 = \frac{\eta_{56}^1}{\eta_{33}^1}(c_{35}^1),^* \ c_{54}^1 = \frac{\eta_{56}^1}{\eta_{44}^1}(c_{45}^1)^*. \tag{2.26}$$

Choosing now for the remaining free elements $c_{\tau\tau'}^s$ and $\eta_{\tau\tau'}^s$ the values equal, e. g., to

$$c_{13}^0 = c_{24}^0 = 1, \ \ c_{35}^1 = c_{45}^1 = \frac{1}{\sqrt{2}}, \tag{2.27}$$

$$\eta_{11}^0 = -\eta_{22}^0 = \eta_{33}^0 = -\eta_{44}^0 = -\eta_{56}^1 = -\eta_{65}^1 = 1, \tag{2.28}$$

we obtain the RWE with the spin blocks of the matrices Γ_4 and η, being equal

$$C^0 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \ C^1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}, \tag{2.29}$$

$$\eta^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \ \eta^1 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \quad (2.30)$$

The RWE constructed in this way with using the Gel'fand – Yaglom basis satisfies the conditions of invariance with respect to the full Lorentz group and of a possibility of its derivation from the invariant Lagrangian function. From the formal point of view, this RWE describes a microobject with nonzero mass and the spins 0, 1. The minimal equations for the spin blocks C^0, C^1 and the whole matrix Γ_4 have the equal form

$$(C^0)^2 - 1 = 0, \ (C^1)^2 - 1 = 0, \ \Gamma_4^2 - 1 = 0, \quad (2.31)$$

from where it follows that the present RWE belongs to the Dirac type with the algebra (2.4). The presence of the multiple roots ± 1 in the blocks C^0, C^1 implies the presence of an additional (besides spin) internal degree of freedom.

We note that the choice of elements (2.27) of the matrix Γ_4 is not unique as long as the derivation of a Dirac-like equation is concerned. In general, to satisfy the characteristic equations (2.31) it is sufficient to demand the condition

$$\Gamma_\mu \Gamma_\nu + \Gamma_\nu \Gamma_\mu = 2\delta_{\mu\nu}. \quad (2.32)$$

It is obvious that only by changing the signs of the numbers c^0_{13}, c^0_{24}, c^1_{35}, c^1_{45} one can define the spin C^0, C^1 in 16 different ways. However, all these (and other possible) variants are related to each other by similarity transformations, and therefore all of them are physically equivalent.

Thus, on the basis of the linking scheme (2.23) one can construct the only (up to a similarity transformation) Dirac-like RWE, which is nondisintegrating in the sense of the full Lorentz group and admitting the Lagrangian formulation.

The above formulated algebraic and group-theoretical substantiation of the dynamical equivalence of the classical Dirac equation and the *SU(2, 2)*-invariant Dirac theory is not yet sufficient for a geometrized description of internal degrees of freedom of the Dirac particles in terms of tensor fields. To consistently realize a possibility of such a description one has to demonstrate that this correspondence persists on the quantum level, which is equivalent to proving a possibility of quantization of the DK field in the Fermi – Dirac statistics.

It might seem that such an assumption contradicts to the famous Pauli's theorem about the relation between spin and statistics [17]. However, this is not quite true. Already in the papers [18; 19] it was shown on examples of the simplest equations for particles with integer and half-integer spins that by using an indefinite metric in the Hilbert space of states it is in principle possible to quantize fields by the anomalous statistics (half-integer spins by the Bose – Einstein statistics and integer spins by the Fermi – Dirac statistics). However, in doing so one obtains unremovable negative probabilities.

An essentially different situation arises in the case of field systems with additional degrees of freedom corresponding to noncompact internal symmetry groups. In such theories, there exist additional conservation laws (ban rules), forbidding transitions which are described by negative probabilities emerging in the quantization with an indefinite metric. Let us consider this question in more detail with regard to the DK equation [20; 21].

With the help of substitution.

$$\Psi(x) = \Psi(\rho) e^{i\rho_\mu x_\mu} \quad (2.33)$$

we proceed from the matrix form of the DK equation in the position representation to the momentum representation

$$(\hat{\rho} + m)\Psi(\rho) = 0, \qquad (2.34)$$

where

$$\hat{\rho} = i\rho_\mu \Gamma_\mu \qquad (2.35)$$

is an operator of the 4-momentum.

As it follows from (2.31), the spin blocks C^0, C^1 contain the only – up to a sign – nonzero root ± 1. A presence of the internal degree of freedom is expressed in this case in the twofold multiplicity of this nonzero root in the characteristic equations of these blocks. Thus, along with the usual 4-momentum operators (2.35), the spin square operator

$$\hat{S}^2 = -[(J^{[12]})^2 + (J^{[23]})^2 + (J^{[31]})^2] \qquad (2.36)$$

and the spin projection operator

$$\hat{S}_n = -i\varepsilon_{ijk} n_i J^{[jk]}, \qquad (2.37)$$

where $J^{[\mu\nu]} = \frac{1}{4}(\Gamma_{[\mu}\Gamma_{\nu]} + \Gamma_{[\mu}^{?}\Gamma_{\nu]}^{?})$, we can assign to this degree of freedom (let us call it *П*-parity for concreteness) an operator $\hat{\Pi}$, which commutes with the above quoted operators and forms together with them the full set of variables for the DK field. Additionally, we complement this assignment by the natural requirements of the diagonalizability of this operator and the real-valuedness of its eigenvalues, and – in analogy with the operators $\hat{S}^2$, $\hat{S}_n$ – by the property

$$\hat{\Pi}\eta = \eta\hat{\Pi}^+. \qquad (2.38)$$

It is not difficult to see that the relativistically invariant definition of the *П*-parity operator, obeying the formulated conditions, has the form

$$\hat{\Pi} = \frac{\rho_\mu \Gamma_\mu^{?}}{im}; \qquad (2.39)$$

and, in particular, in the rest frame

$$\hat{\Pi}_0 = \Gamma_4^{?}. \qquad (2.40)$$

The eigenvalues of $\hat{\Pi}$ we will denote by $\lambda_i, i = 1, 2$. (In the rest frame it holds $\lambda_1 = 1, \lambda_2 = -1$).

In the second quantization, the sign factors of the energy and the charge densities of a classical field system acquire an important role. The presence of the spin spectrum and the *П*-parity infers that the signs of these quantities can depend not only on the mass sign (which is meant to be the sign of the matrix Γ_4 eigenvalues distinguishing between the positive and negative frequency solutions of the equation (2.34)), but also on the quantum numbers *i* and *s*.

In other words, both the energy and the charge in such theories appear to be, generally speaking, indefinite. This circumstance is conveniently reflected in the variable $g_{is}^{(\pm)}$, whose values, evaluated in the rest frame, specify the sign of the energy density in the state $\psi_{is}^{(\pm)}$. Computing $g_{is}^{(\pm)}$ for the DK equation yields [22]:

$$g_{1s}^{(+)} = g_{2s}^{(-)} = 1, \; g_{1s}^{(-)} = g_{2s}^{(+)} = -1. \qquad (2.41)$$

Now we can directly implement the quantization. Representing the operator wave-fuctions Ψ, $\bar{\Psi}$ by the series

$$\Psi(x) = \frac{1}{(2\pi)^{3/2}} \sum_i \sum_s \left[a_{is}(p)\psi_{is}^{(+)}(p)e^{ipx} + b_{is}^+(p)\psi_{is}^{(-)}(p)e^{-ipx}\right] d^3p, \qquad (2.42)$$

$$\bar{\Psi}(x) = \frac{1}{(2\pi)^{3/2}} \sum_i \sum_s \left[a_{is}^+(p)\bar{\psi}_{is}^{(+)}(p)e^{-ipx} + b_{is}(p)\bar{\psi}_{is}^{(-)}(p)e^{ipx}\right] d^3p, \qquad (2.43)$$

we postulate the commutation relations for the annihilation and creation operators

$$[a_{is}(p), a^{+}_{i's'}(p')]_{+} = g^{(+)}_{is}\delta_{ii'}\delta_{ss'}\delta(p-p'), \tag{2.44}$$

$$[b_{is}(p), b^{+}_{i's'}(p')]_{+} = -g^{(-)}_{is}\delta_{ii'}\delta_{ss'}\delta(p-p'), \tag{2.45}$$

(no summation over the indices *i* and *s* here; all the other anticommutators identically vanish), which correspond to the quantization of the DK field in the Fermi – Dirac statistics.

The particle and antiparticle number operators leading to the correct eigenvalues are defined in the following way:

$$N^{(+)}_{is} = g^{(+)}_{is} a^{+}_{is} a_{is}, \; N^{(-)}_{is} = -g^{(-)}_{is} b^{+}_{is} b_{is}. \tag{2.46}$$

Inserting the series (2.42), (2.43) into the expressions for the energy and the charge operators yields

$$E = \int \{(\partial_4 \bar{\Psi})\Gamma_4 \Psi - \bar{\Psi}\Gamma_4 \partial_4 \Psi\} d^3x, \tag{2.47}$$

$$Q = e\int \bar{\Psi}\Gamma_4 \Psi \, d^3x. \tag{2.48}$$

Taking into account the relations (2.44) – (2.46) and the normalization by charge

$$\bar{\Psi}\Gamma_4\Psi = \pm 1, \tag{2.49}$$

we obtain the final expressions for the operators E and Q

$$E = \sum_i \sum_s \left(N^{(+)}_{is}\varepsilon^{(+)}_{is} + N^{(-)}_{is}\varepsilon^{(-)}_{is}\right), \tag{2.50}$$

$$Q = e\sum_i \sum_s \left(N^{(+)}_{is} - N^{(-)}_{is}\right), \tag{2.51}$$

where $\varepsilon^{(\pm)}_{is} = |p_0|$, and the indices of $\varepsilon^{(\pm)}_{is}$ indicate the relation to the corresponding state.

The expressions (2.50), (2.51) mean that the anticommutation relations (2.44), (2.45) ensure a correct corpuscular picture of the field. Moreover, it is easy to check that they lead to causal commutation relations for the field operators [22].

Since the right-hand sides of some quantization conditions (2.44), (2.45) contain the «wrong» minus sign, the corresponding state vectors must have a negative valued norm. In other words, a quantum description of the DK field in the Fermi – Dirac statistics implies using the space of states H with the indefinite metrics

$$H = H_{+} \oplus H_{-}, \tag{2.52}$$

where H_{+} and H_{-} are the subspaces with positive and negative state vector norms, respectively. In the considered case the subspaces H_{+} and H_{-} are spanned by the states

$$H_{+}: \left(\prod_{N_1} a^{+}_{1s}\right)\left(\prod_{N_2} b^{+}_{2s}\right)\left(\prod_{N_3} a^{+}_{2s}\right)\left(\prod_{N_4} b^{+}_{1s}\right)|0\rangle; \tag{2.53}$$

$$H_{-}: \left(\prod_{N_5} a^{+}_{1s}\right)\left(\prod_{N_6} b^{+}_{2s}\right)\left(\prod_{N_7} a^{+}_{2s}\right)\left(\prod_{N_8} b^{+}_{1s}\right)|0\rangle. \tag{2.54}$$

Here N_1, N_2, N_5, and N_6 are arbitrary nonnegative integers, (N_3+N_4) is an even number, and (N_7+N_8) is an odd number. For single particle states the partitions (2.53), (2.54) acquire the form

$$H_{+}: \Psi^{(+)}_{1s}, \Psi^{(-)}_{1s}, \; H_{-}: \Psi^{(+)}_{2s}, \Psi^{(-)}_{2s}, \tag{2.55}$$

$$H_{+}: \Psi^{(+)}_{1s}, \Psi^{(-)}_{2s}, \; H_{-}: \Psi^{(+)}_{2s}, \Psi^{(-)}_{1s}. \tag{2.56}$$

Additionally, for a correct probabilistic interpretation of the theory it is necessary to ensure the absence of transitions between the states of H_{+} and H_{-} by including interactions. Let us show that such transitions are indeed forbidden.

Consider the Lagrangian

$$\mathcal{L} = -\overline{\Psi}(x)\big(\Gamma_\mu \partial_\mu + m\big)\Psi(x) + \mathcal{L}_{int}, \tag{2.57}$$

where $\mathcal{L}_{int}$ is an interaction Lagrangian which does not violate an internal symmetry inherent to a free field. For instance, in the case of electromagnetic interaction it reads

$$\mathcal{L}_{int} = e\overline{\Psi}\Gamma_\mu A_\mu \Psi + \overline{\Psi} F_{\mu\nu}\Gamma_{[\mu}\Gamma_{\nu]}\Psi. \tag{2.58}$$

It is evident that the operator $\widehat{\Pi}$ (2.39) belongs to the transformations of the internal symmetry group of the (2.57), (2.58) (compare (2.39) with the generators (2.10) of this group). The invariance of the quoted Lagrangian under the transformations

$$\Psi \to e^{i\widehat{\Pi}\theta}\Psi \tag{2.59}$$

leads to the conserved «charge»

$$G \sim \int \overline{\Psi}(x)\Gamma_4 \widehat{\Pi}\Psi(x)\, d^3x. \tag{2.60}$$

The charge *G* can be transformed to the form

$$G \sim \sum_i \sum_s \lambda_i \Big(N_{is}^{(+)} - N_{is}^{(-)}\Big) = \sum_s (N_{1s}^{(+)} - N_{2s}^{(+)} - N_{1s}^{(-)} + N_{2s}^{(-)}), \tag{2.61}$$

where it is accounted that$\lambda_1 = 1,\ \lambda_2 = -1$. For the sake of convenience we also rewrite the formula (2.51) in the expanded form

$$Q \sim \sum_s (N_{1s}^{(+)} + N_{2s}^{(+)} - N_{1s}^{(-)} - N_{2s}^{(-)}). \tag{2.62}$$

Comparing the partitions (2.55), (2.56) with the expressions (2.61), (2.62), we draw the conclusion that to the single-particle states belonging to the subspaces H_+ and H_- correspond the following signs of the charges *Q* and *G*:

$$H_+ : (1,1), (-1,1), \quad H_- : (1,-1), (-1,-1) \tag{2.63}$$

(the first number in the parenthesis corresponds to the electric charge *Q*, while the second number corresponds to the additional charge *G*).

From it follows (2.63) that a simultaneous fulfillment of the conservation laws for the charges *Q* and *G* leads to a prohibition of physically inappropriate transitions between states from the subspaces with the positive and the negative norms of state vectors.

We remark that if instead of the continuous transformations (2.59) we consider the discrete transformations

$$\Psi_+ \to \Psi_+,\ \Psi_- \to -\Psi_-, \tag{2.64}$$

they will be reduced to these:

$$a_{1s}, a_{1s}^+ \to a_{1s}, a_{1s}^+, \qquad b_{1s}, b_{1s}^+ \to b_{1s}, b_{1s}^+, \tag{2.65}$$

$$a_{2s}, a_{1s}^+ \to -a_{2s}, -a_{2s}^+, \qquad b_{2s}, b_{2s}^+ \to -b_{2s}, -b_{2s}^+.$$

In mathematical literature this operation bears the name of a canonical, or *J*-symmetry. It underpins the theory of linear operators in spaces with indefinite metrics, which are also called Hilbert spaces with *J*-metrics, or the Krein spaces. The *J*-symmetry corresponds to the superselection operator forbidding transitions from H_+ to H_-, which is in agreement with our result established above. Upon localizing the internal symmetry group, which has a spatio-temporal origin, and considering the corresponding gauge theory the discrete *J*-symmetry also helps to exclude transitions featuring negative probabilities.

Thus, the considered quantization procedure of the DK equation by the Fermi – Dirac statistics appears to be correct also from the point of view of a probabilistic interpretation of the theory. This fact in combination with other algebraic and group-theoretical properties discussed above substantiates a possibility in principle to describe internal degrees of freedom of the Dirac particles in the geometrized approach.

Conclusion

Let us list once again in a concise form the presented results. On the basis of the use of extended sets of irreducible representations of the Lorentz group, it is given:

• semiphenomenological description of the internal structure of microobjects with lower spins;

• a description of the isospin degrees of freedom, in particular the chirality of massive microobjects, by means of the RWE that does not decay in the Lorentz group and which has internal symmetry of geometric origin;

• a joint description of massless fields with helicities as a single physical object, on this basis the possibility of a semiphenomenological description of the interaction of strings and membranes in Minkowski space is shown;

• matrix interpretation of the mechanism of mass generation of vector fields, which differs from the well-known Higgs mechanism and does not lead to the appearance of additional scalar or any other massive particles;

• matrix interpretation of massive gauge-invariant fields in the approach of the RWE theory;

• finally, a non-disintegrating RWE is described that describes a massively massless vector field with three types of massive and one massless quanta. This field may well be interpreted as an electroweak field. The need for the appearance of a scalar massive field is justified in a completely new way. It turns out that in the approach of the theory of RWE, the indicated vector field can exist only in a «bundle» with a massive scalar field, forming together with it a single unified physical object. Otherwise, the free massive and massless vector fields appear as independent ones, i. e. are unconnected in a relativistically invariant sense by equations.

The novelty and the possibility of applying the results obtained are as follows. Global unitary symmetries, which are used in modern gauge models of fundamental particles and their interactions, are based on non-geometric origin. In other words, in these models the simplest Dirac equation is taken as the initial one, the free function of which is «hung» by the free non-Lorentz index. Thus, the relationship between the properties of space-time and the material world is manifested only after the localization of these symmetries. Our proposal is to rely on internal symmetries in local-calibration models, which already in the original global version have a geometric origin, i.e. are inherent in equations that do not decay over the full Lorentz group. The Dirac – Kähler equation and its algebraic generalizations considered in our paper can serve as a possible candidate for the role of such RWE.

This approach, in our opinion, provides a closer relationship between space-time and the material world. In addition, the expansion of the class of basic RWE should lead to new physical effects and, possibly, eliminate some of the difficulties that occur in the Standard Model and the theory of superstrings.

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