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A PRIORI CHOICE OF THE REGULARIZATION PARAMETER IN AN ITERATIVE PROCEDURE OF AN EXPLICIT TYPE SOLUTION OF LINEAR ILL-POSED EQUATIONS

The explicit iteration procedure with variable-alternating step for solution of ill-posed operator equations of the first kind is proposed in Hilbert space. Convergence of a method is proved in case of a priori choice of the regularization parameter in usual norm of Hilbert space, supposing that not only the right part of the equation but the operator as well have errors. The estimations of an error and a priori stopping moment are received. The results given in the article can be used in theoretical research in solving linear operator equations, as well as in solving applied ill-posed problems.

Key words: *explicit iteration procedure, ill-posed problem, Hilbert space, self-adjoint and non-self-adjoint approximately given operator, operator equation of the first kind, a priori stopping moment.*

Априорный выбор параметра регуляризации в итерационной процедуре явного типа решения линейных некорректных уравнений

В гильбертовом пространстве предлагается явная итерационная процедура с попеременно чередующимся шагом решения некорректных операторных уравнений первого рода. Доказана сходимость метода в случае априорного выбора параметра регуляризации в исходной норме гильбертова пространства в предположении, что погрешности имеются не только в правой части уравнения, но и в операторе. Получены оценки погрешности метода и априорный момент останова. Приведенные результаты могут быть использованы в теоретических исследованиях при решении линейных операторных уравнений, а также при решении прикладных некорректных задач.

Ключевые слова: *явная итерационная процедура, некорректная задача, гильбертово пространство, самосопряженный и несамосопряженный приближенно заданный оператор, операторное уравнение первого рода; априорный момент останова.*

Introduction

There is a large class of problems where solutions are unstable to small changes in the source data, i. e. arbitrarily small changes in the source data can lead to large changes in solutions. Tasks of this type belong to the class of incorrect tasks.

A significant part of the problems encountered in applied mathematics, physics, engineering and management can be represented in the form of an operator equation of the first kind

$$Ax = y, \quad x \in X, \quad y \in Y \tag{1}$$

with the specified operator $A: X \to Y$ and an element y, where X and Y – metric spaces, and in specially specified cases – Banach or even Hilbert spaces.

J. Hadamard [1] introduced the following concept of correctness:

Definition. *The task of finding a solution* $x \in X$ *equation* (1) *are called correct (or correctly posed, or Hadamard-correct) if, for any fixed right-hand side of equation* (1), *its solution is* $y = y_0 \in Y$ *equation* (1) *its solution*:

a) exists in space X;

b) it is uniquely defined in the space X;

c) it is stable in the space X, i.e. it continuously depends on the right side of $y \in Y$. *In case of violation of any of these conditions, the task is called* <u>*incorrect*</u> *(*<u>*incorrectly posed*</u>*); more specifically, in case of violation of condition c), it is called* <u>*unstable*</u>.

It can be seen from the definition that Hadamard correctness is equivalent to unambiguous certainty and continuity of the inverse operator A^{-1} over the entire space *Y*.

For many years, it has been believed in mathematics that only correct problems have the right to exist, that only they correctly reflect the real world.

There is an opinion about incorrect tasks that they do not have a physical reality, so their solution is meaningless. As a result, incorrect tasks have not been studied for a long time. However, in practice, the need to solve incorrect tasks has become more and more frequent and persistent.

Such problems include the Cauchy problem for the Laplace equation, the problem of solving an integral equation of the first kind, the problem of differentiating a function given approximately, the numerical summation of Fourier series when the coefficients are known approximately in the metric l_2, the inverse problem of gravimetry, the inverse problem of potential theory, the problem of spectroscopy, etc.

Iterative methods occupy a special place among the methods of solving incorrect problems, since they are easily implemented on a PC. Various iterative schemes for solving incorrectly set tasks have been proposed in the works [2–12].

In this article, an explicit iterative procedure with alternating steps for solving ill-posed problems in Hilbert space is proposed and some of its properties are investigated.

Comparison of the proposed method with the well-known *explicit Landweber iteration method* [2] $x_{n+1,\delta} = x_{n,\delta} + \alpha\left(y_\delta - Ax_{n,\delta}\right)$, $x_{0,\delta} = 0$ shows that the orders of their optimal estimates are the same.

The advantage of explicit methods is that explicit methods do not require operator inversion, but only require calculating operator values on successive approximations. In the *Landweber method*, the parameter α (anti-gradient step) is constrained from above – the inequality $0 < \alpha \le \frac{5}{4\|A\|}$, which may lead in practice to the need for a large number of iterations. However, the proposed method has an advantage over the *Landweber method* in the following: to achieve optimal accuracy, it will require making the number of iterations about 3 times less than the iteration method [2].

As is known, the error of the method of simple iteration with a constant [2–4] or a variable [8] step depends on the sum of the anti-gradient steps, and moreover, in such a way that in order to reduce the number of operations, it is desirable that the anti-gradient steps be as large as possible.

However, these steps are subject to restrictions from above [2–4; 8]. The idea arises to try to loosen these restrictions.

This was done by choosing two values for the step α and β alternately, where β it is no longer required to meet the previous requirements.

The explicit iterative method considered in the article will find practical application in applied mathematics: it can be used to solve problems encountered in optimal control theory, mathematical economics, geophysics, potential theory, antenna synthesis, acoustics, plasma diagnostics, in terrestrial or aerial geological exploration, in solving the inverse kinematic problem of seismics, space research (spectroscopy) and medicine (computed tomography).

1. Setting the task

An equation of the first kind is solved in a real Hilbert space H

$$Ax = y \tag{2}$$

with a positive bounded self-adjoint operator A for which zero is not an eigenvalue, but $0 \in Sp(A)$, therefore, the problem under consideration is incorrect. It is assumed that for the exact right-hand side of y, equation (2) has a single exact solution x^*. To find it, we use an explicit iterative procedure with alternating steps:

$$x_{n+1} = \left(I - \alpha_{n+1}A\right)x_n + \alpha_{n+1}y, \quad x_0 = 0,\ \alpha_{2n+1} = \alpha,\ \alpha_{2n+2} = \beta,\ n = 0, 1, 2, \dots, \tag{3}$$

where I is the identical operator.

In the case of the approximate right-hand side y_δ, $\|y - y_\delta\| \le \delta$, iterations (3) will take the form:

$$x_{n+1,\delta} = \left(I - \alpha_{n+1}A\right)x_{n,\delta} + \alpha_{n+1}y_\delta, \quad x_{0,\delta} = 0,\ \alpha_{2n+1} = \alpha,\ \alpha_{2n+2} = \beta,\ n = 0, 1, 2, \dots. \tag{4}$$

Below, the convergence of method (4) means the statement that approximations (4) come arbitrarily close to the exact solution of equation (2) with a suitable choice of n and sufficiently small ones δ. In other words, the iterative method (4) is convergent if $\lim\limits_{\delta\to 0}\left(\inf\limits_n \left\|x - x_{n,\delta}\right\|\right) = 0$ [9].

Next, we will assume that $\|A\| = 1$.

2. A priori choice of the number of iterations with the approximate right-hand side

Methods (3) and (4) were considered in [6], in which the convergence of both methods in the initial norm of Hilbert space was studied. For their convergence in [6], it is required that when $0 < \alpha < 2,\ \beta > 0$:

$$\left|(1 - \alpha\lambda)(1 - \beta\lambda)\right| < 1 \tag{5}$$

for anyone $\lambda \in (0,1]$. Condition (5) is equivalent to the combination of two conditions:

$$(\alpha + \beta)^2 < 8\alpha\beta, \tag{6}$$

$$\alpha\beta < \alpha + \beta. \tag{7}$$

For method (4), the priori choice of the number of iterations is studied [6]. It is proved that the iterative process (4) converges under the conditions (6), (7) and $0 < \alpha < 2$, if we choose the number of iterations n depending on δ so that $n\delta \to 0,\ n \to \infty, \delta \to 0.$

Assuming that the exact solution of x^* is originally representable, i. e. $x^* = A^s z, s > 0$ and under the conditions $0 < \alpha < 2$, (6),

$$\alpha + \beta < \frac{3}{2}\alpha\beta, \tag{8}$$

$$\frac{1}{16} + \alpha\beta \le \alpha + \beta. \tag{9}$$

The following estimation of the error of the method (4) is obtained:

$$\left\|x - x_{n,\delta}\right\| \le s^s \left[n(\alpha+\beta)\right]^{-s} \|z\| + \frac{n}{2}(\alpha+\beta)\delta. \qquad (10)$$

To find the optimal error estimate for n, we equate the derivative of n from the right side of inequality (10) to zero.

Then the optimal error estimate for n has the form

$$\left\|x - x_{n,\delta}\right\|_{\text{opt}} \le (1+s)\, 2^{-s/(s+1)} \delta^{s/(s+1)} \|z\|^{1/(s+1)}$$

and it turns out when $n_{\text{opt}} = s\left(\frac{\alpha+\beta}{2}\right)^{-1} 2^{-s/(s+1)} \delta^{-1/(s+1)} \|z\|^{1/(s+1)}$.

Thus, the optimal estimate for method (4) with an inaccuracy on the right side of the equation turns out to be the same as the estimate for the simple iteration method [2].

Therefore, method (4) does not provide an advantage in majority estimates compared to method [2].

But it gives a win in the following: in the method of simple iteration with a constant step [2] $0 < \alpha \le \frac{5}{4}$, and in the method (4) $0 < \frac{\alpha+\beta}{2} < 4$ [6].

Therefore, by choosing α and β alternating accordingly, it is possible to make the n_{opt} in method (4) about three times smaller than in the method of simple iteration with a constant step [2].

Thus, using method (4), to achieve optimal accuracy, it is enough to make iterations three times less than using method [2]. We present several suitable values satisfying the required conditions:

α	0,8	0,9	1,0	1,1	1,15	1,17	1,3
β	4,4	5,0	5,5	6,1	6,4	6,5	4,1

The largest amount $\alpha+\beta$ and, therefore, the largest gain in the amount of calculations are given by the values $\alpha = 1.17$ and $\beta = 6.5$.

Since in the highlighted case $\alpha+\beta$=7.67, the condition $\alpha+\beta<8$ shows that almost the maximum possible gain has been achieved.

Remark 1. *Convergence estimates were obtained for the case when* $l = m = \frac{n}{2}$. *In the case when* $l = m+1$, *in all assessments the* $\frac{n(\alpha+\beta)}{2}$ *should be replaced by* $l\alpha + m\beta$.

Remark 2. *We believe that* $\|A\| = 1$. *In fact, all the results are easily transferred to the case when* $\|A\| < \infty$.

3. A priori choice of the regularization parameter with an approximate operator

We prove the convergence of method (4) in the case of an a priori choice of the regularization parameter when solving an equation $A_\eta x = y_\delta$ with an approximately given operator A_η and an approximate right-hand side y_δ, we obtain a priori error estimates.

Similar questions were studied in [3; 6–7], but only for other methods.

3.1. The case of self-adjoint operators

Let $A = A^* \geq 0$, $A_\eta = A_\eta^* \geq 0$, $Sp(A_\eta) \subseteq [0, 1]$, $0 < \eta \leq \eta_0$. The iterative method (4) is written as

$$x_{n(\eta,\delta)} = g_n(A_\eta) y_\delta, \tag{4^1}$$

where $g_n(\lambda) = \lambda^{-1}\left[1-(1-\alpha\lambda)^{n/2}(1-\beta\lambda)^{n/2}\right]$. In the work [6] at $0 < \alpha < 2$, (6) and (7) the conditions for the functions $g_n(\lambda)$ are obtained:

$$\sup_{0\leq\lambda\leq 1} |g_n(\lambda)| \leq \gamma n, \quad \gamma = \frac{\alpha+\beta}{2}, \quad n > 0, \tag{11}$$

$$\sup_{0\leq\lambda\leq 1} |1-\lambda g_n(\lambda)| \leq \gamma_0, \quad \gamma_0 = 1, \quad n > 0. \tag{12}$$

Assuming that the exact solution of x^* equation (2) is source–representable, i. e. $x^* = A^s z, s > 0$, where $s > 0$ is the degree of source-representability of the exact solution, $\|z\| \leq \rho$ and under the conditions $0 < \alpha < 2$, (6), (8) and (9) the estimation was received:

$$\sup_{0\leq\lambda\leq 1} \lambda^s |1-\lambda g_n(\lambda)| \leq \gamma_s n^{-s}, \ \gamma_s = \left(\frac{s}{\alpha+\beta}\right)^s, (n > 0), \ 0 < s < \infty. \tag{13}$$

Fair

Lemma 1. *Let* $A = A^* \geq 0$, $A_\eta = A_\eta^* \geq 0$, $\|A_\eta - A\| \leq \eta$, $Sp(A_\eta) \subseteq [0, 1]$, $(0 < \eta \leq \eta_0)$ *and the conditions are met* $0 < \alpha < 2$, (6), (7), (12). *Then* $\|G_{n\eta} v\| \to 0$ *by* $n \to \infty$, $\eta \to 0$, $\forall v \in H$, *where* $G_{n\eta} = I - A_\eta g_n(A_\eta)$.

Proof.

We have

$$\|G_{n\eta} v\| = \|(I - A_\eta g_n(A_\eta))v\| = \left\|\int_0^1 (1-\lambda g_n(\lambda))\, dE_\lambda v\right\| =$$

$$= \left\|\int_0^1 (1-\alpha\lambda)^{n/2}(1-\beta\lambda)^{n/2}\, dE_\lambda v\right\| \leq \left\|\int_0^\varepsilon (1-\alpha\lambda)^{n/2}(1-\beta\lambda)^{n/2}\, dE_\lambda v\right\| + \left\|\int_\varepsilon^1 (1-\alpha\lambda)^{n/2}(1-\beta\lambda)^{n/2}\, dE_\lambda v\right\|.$$

$$\left\|\int_\varepsilon^1 (1-\alpha\lambda)^{n/2}(1-\beta\lambda)^{n/2}\, dE_\lambda v\right\| \leq q^{n/2}(\varepsilon)\left\|\int_\varepsilon^1 dE_\lambda v\right\| \to 0, \ n \to \infty,$$

since for $\lambda \in [\varepsilon, 1]$ $|(1-\alpha\lambda)(1-\beta\lambda)| \leq q(\varepsilon) < 1$.

$$\left\|\int_0^\varepsilon (1-\alpha\lambda)^{n/2}(1-\beta\lambda)^{n/2}\, dE_\lambda v\right\| \leq \left\|\int_0^\varepsilon dE_\lambda v\right\| = \|E_\varepsilon v\| \to 0, \varepsilon \to 0$$

due to the properties of the spectral function [5–6]. Therefore, $\|G_{n\eta} v\| \to 0$ by $n \to \infty$, $\eta \to 0$. Lemma 1 is proved.

The convergence condition for method (4) gives

Theorem 1. *Let* $A = A^* \ge 0$, $A_\eta = A_\eta^* \ge 0$, $\|A_\eta - A\| \le \eta$, $Sp(A_\eta) \subseteq [0,1]$, $(0 < \eta \le \eta_0)$, $y \in R(A)$, $\|y - y_\delta\| \le \delta$ *and the conditions are met* $0 < \alpha < 2$, (6), (7), (11). *Lets choose the parameter* $n = n(\delta,\eta)$ *in approximation* (4) *so that* $(\delta+\eta)\,n(\delta,\eta) \to 0$ *by* $n(\delta,\eta) \to \infty$, $\delta \to 0, \eta \to 0$. *Then* $x_{n(\delta,\eta)} \to x^*$ *by* $\delta \to 0, \eta \to 0$.

Proof.

From (4^1) we have $x_n = g_n(A_\eta)y_\delta$. Then

$$x_n - x^* = g_n(A_\eta)y_\delta - x^* = -G_{n\eta}x^* + G_{n\eta}x^* + g_n(A_\eta)y_\delta - x^* =$$
$$= -G_{n\eta}x^* + (I - A_\eta g_n(A_\eta))x^* + g_n(A_\eta)y_\delta - x^* = -G_{n\eta}x^* + g_n(A_\eta)(y_\delta - A_\eta x^*).$$

Therefore, $x_n - x^* = -G_{n\eta}x^* + g_n(A_\eta)(y_\delta - A_\eta x^*)$.

Because according to the condition (11) $\|g_n(A_\eta)\| \le \sup\limits_{0\le\lambda\le 1} |g_n(\lambda)| \le \gamma n$, $\gamma = \dfrac{\alpha+\beta}{2}$, and $\|y_\delta - A_\eta x^*\| \le \|y_\delta - y\| + \|y - A_\eta x^*\| = \|y_\delta - y\| + \|Ax^* - A_\eta x^*\| \le \delta + \|A - A_\eta\|\|x^*\| \le \delta + \eta\|x^*\|$, then we have $\|x_{n(\delta,\eta)} - x^*\| \le \|G_{n\eta}x^*\| + \|g_n(A_\eta)(y_\delta - A_\eta x^*)\| \le \|G_{n\eta}x^*\| + \gamma n(\delta + \eta\|x^*\|)$.

It follows from Lemma 1 that $\|G_{n\eta}x^*\| \to 0$ *by* $n \to \infty$, $\eta \to 0$, and by the condition of theorem 1 $n(\delta+\eta) \to 0$ by $\delta \to 0$, $\eta \to 0$. Thus, $\|x_{n(\delta,\eta)} - x^*\| \to 0$, $\delta \to 0, \eta \to 0$.

Theorem 1 has been proved.

Theorem 2. *Let* $A = A^* \ge 0$, $A_\eta = A_\eta^* \ge 0$, $\|A_\eta - A\| \le \eta$, $Sp(A_\eta) \subseteq [0,1]$, $(0 < \eta \le \eta_0)$, $y \in R(A)$, $\|y - y_\delta\| \le \delta$ *and the conditions are met* $0 < \alpha < 2$, (6), (8), (9), (11), (12), (13). *If the exact solution is representable from the source, i.e.* $x^* = A^s z$, $s > 0$, $\|z\| \le \rho$, *then the error estimate is fair:*

$$\|x_{n(\delta,\eta)} - x^*\| \le \gamma_0 c_s \eta^{\min(1,s)}\rho + \gamma_s n^{-s}\rho + \gamma n(\delta + \eta\|x^*\|),\ 0 < s < \infty.$$

Proof.

We have, using the source-like representability of the exact solution,

$$\|G_{n\eta}x^*\| = \|G_{n\eta}A^s z\| \le \|G_{n\eta}(A^s - A_\eta^s)z\| + \|G_{n\eta}A_\eta^s z\| \le \gamma_0 c_s \eta^{\min(1,s)}\rho + \gamma_s n^{-s}\rho,$$

since by lemma 1.1 [3, p. 91] $\|A_\eta^s - A^s\| \le c_s \eta^{\min(1,s)}$, $c_s = \text{const}$ ($c_s \le 2$ for $0 < s \le 1$). Then

$$\|x_{n(\delta,\eta)} - x^*\| \le \gamma_0 c_s \eta^{\min(1,s)}\rho + \gamma_s n^{-s}\rho + \gamma n(\delta + \eta\|x^*\|),\ 0 < s < \infty. \tag{14}$$

Theorem 2 has been proved.

If we minimize the right side of the estimate (14) by n, we get the value of an a priori stopping moment: $n_{\text{opt}} = \left[\dfrac{s\gamma_s\rho}{\gamma\left(\delta + \|x^*\|\eta\right)}\right]^{1/(s+1)} = d_s\rho^{1/(s+1)}\left[\delta + \eta\|x^*\|\right]^{-1/(s+1)}$,

where $d_s = \left(\dfrac{s\gamma_s}{\gamma}\right)^{1/(s+1)}$. From here $n_{\text{opt}} = s\left(\dfrac{\alpha+\beta}{2}\right)^{-1} 2^{-s/(s+1)}\rho^{1/(s+1)}\left(\delta + \eta\|x^*\|\right)^{-1/(s+1)}$.

Substitute n_{opt} in the estimate (14), we get

$$\left\|x_{n(\delta,\eta)} - x^*\right\|_{\text{opt}} \le \gamma_0 c_s\eta^{\min(1,s)}\rho + \gamma_s\rho\left(d_s\rho^{1/(s+1)}\right)^{-s}\left(\delta + \eta\|x^*\|\right)^{s/(s+1)} +$$
$$+\gamma\left(\delta + \eta\|x^*\|\right)d_s\rho^{1/(s+1)}\left(\delta + \eta\|x^*\|\right)^{-1/(s+1)} =$$
$$= \gamma_0 c_s\eta^{\min(1,s)}\rho + \left(\delta + \eta\|x^*\|\right)^{s/(s+1)}\left(d_s^{-s}\gamma_s\rho^{1/(s+1)} + \gamma d_s\rho^{1/(s+1)}\right) =$$
$$= \gamma_0 c_s\eta^{\min(1,s)}\rho + \rho^{1/(s+1)}c_s'\left(\delta + \eta\|x^*\|\right)^{s/(s+1)},$$

where $c_s' = d_s^{-s}\gamma_s + \gamma d_s = \left(s^{1/(s+1)} + s^{-s/(s+1)}\right)\gamma^{s/(s+1)}\gamma_s^{1/(s+1)} = (1+s)2^{-s/(s+1)}$. From here

$$\left\|x_{n(\delta,\eta)} - x^*\right\|_{\text{opt}} \le c_s\eta^{\min(1,s)}\rho + (1+s)2^{-s/(s+1)}\rho^{1/(s+1)}\left(\delta + \eta\|x^*\|\right)^{s/(s+1)}.$$

3.2. The case of non-self-adjoint operators

In the case of a non-self-adjoint problem, the iterative method (4) will take the form:

$$x_{(n+1)(\eta,\delta)} = x_{n,\delta} - \alpha_{n+1}\left(A_\eta^*A_\eta x_{n(\eta,\delta)} - A_\eta^* y_\delta\right), \quad x_{0(\eta,\delta)} = 0,$$
$$\alpha_{2n+1} = \alpha, \quad n = 0, 1, 2, \ldots, \ \alpha_{2n+2} = \beta, \quad n = 0, 1, 2, \ldots. \tag{15}$$

It can be written like this:

$$x_{n(\eta,\delta)} = g_n(A_\eta^*A_\eta)A_\eta^* y_\delta. \tag{16}$$

Lemma 1 implies.

Lemma 2. *Let* $\|A_\eta - A\| \le \eta$, $\|A_\eta\|^2 \le 1$, $(0 < \eta \le \eta_0)$ *and the conditions are met* $0 < \alpha < 2$, (6), (7), (12). Then

$$\|K_{n\eta}v\| \to 0 \ \textit{by} \ n \to \infty, \quad \eta \to 0, \ \forall v \in N(A)^\perp = \overline{R(A^*)}, \tag{17}$$

$$\|\tilde{K}_{n\eta}z\| \to 0 \ \textit{by} \ n \to \infty, \quad \eta \to 0, \ \forall z \in N\left(A^*\right)^\perp = \overline{R(A)}, \tag{18}$$

where $K_{n\eta} = I - A_\eta^*A_\eta g_n\left(A_\eta^*A_\eta\right)$, $\tilde{K}_{n\eta} = I - A_\eta A_\eta^* g_n\left(A_\eta A_\eta^*\right)$.

We use lemma 2 to prove the following theorem.

Theorem 3. *Let* $\|A - A_\eta\| \le \eta$, $\|A_\eta\|^2 \le 1$, $(0 < \eta \le \eta_0)$, $y \in R(A)$, $\|y_\delta - y\| \le \delta$ *and the conditions are met* $0 < \alpha < 2$, (6), (7), (12).). *Let's choose the parameter* $n = n(\delta, \eta)$ *so that*

$$(\delta + \eta)^2 n(\delta, \eta) \to 0 \text{ by } n(\delta, \eta) \to \infty, \delta \to 0, \ \eta \to 0. \tag{19}$$

Then $x_{n(\delta,\eta)} \to x^*$ *by* $\delta \to 0$, $\eta \to 0$.

<u>*Proof.*</u>

For the approximation error $x_{n(\delta,\eta)}$, we have

$$x_{n(\delta,\eta)} - x^* = -K_{n\eta}x^* + g_n(A_\eta^* A_\eta)A_\eta^*\left(y_\delta - A_\eta x^*\right). \tag{20}$$

Here $\left\|g_n\left(A_\eta^* A_\eta\right)A_\eta^*\right\| = \left\|g_n\left(A_\eta^* A_\eta\right)\left(A_\eta^* A_\eta\right)^{1/2}\right\| \le \gamma_* n^{1/2}$, $\gamma_* = \sup\limits_{n>0}\left(n^{-1/2} \sup\limits_{0\le\lambda\le1} \lambda^{1/2}|g_n(\lambda)|\right) \le (\alpha + \beta)^{1/2}$ (see the lemma 3.1 [3, p. 35] and [6]). Because

$$\left\|y_\delta - A_\eta x^*\right\| \le \|y_\delta - y\| + \left\|y - A_\eta x^*\right\| = \|y_\delta - y\| + \left\|Ax^* - A_\eta x^*\right\| \le \delta + \eta\left\|x^*\right\|,$$

that $\left\|g_n\left(A_\eta^* A_\eta\right)A_\eta^*\left(y_\delta - A_\eta x^*\right)\right\| \le (\alpha + \beta)^{1/2} n^{1/2}\left(\delta + \left\|x^*\right\|\eta\right)$. There fore

$$\left\|x_{n(\delta,\eta)} - x^*\right\| \le \left\|K_{n\eta}x^*\right\| + \left\|g_n\left(A_\eta^* A_\eta\right)A_\eta^*\left(y_\delta - A_\eta x^*\right)\right\| \le \left\|K_{n\eta}(x^*)\right\| + (\alpha + \beta)^{1/2} n^{1/2}\left(\delta + \eta\left\|x^*\right\|\right).$$

Let's show that $\left\|K_{n\eta}x^*\right\| \to 0$ by $n \to \infty, \eta \to 0.$ Really,

$$\left\|K_{n\eta}x^*\right\| = \left\|(I - A_\eta^* A_\eta g_n(A_\eta^* A_\eta))x^*\right\| = \left\|\int_0^1 (1 - \lambda g_n(\lambda))\, dE_\lambda x^*\right\| =$$

$$= \left\|\int_0^1 (1 - \alpha\lambda)^{n/2}(1 - \beta\lambda)^{n/2}\, dE_\lambda x^*\right\| \le \left\|\int_0^\varepsilon (1 - \alpha\lambda)^{n/2}(1 - \beta\lambda)^{n/2}\, dE_\lambda x^*\right\| + \left\|\int_\varepsilon^1 (1 - \alpha\lambda)^{n/2}(1 - \beta\lambda)^{n/2}\, dE_\lambda x^*\right\|.$$

Then $\left\|\int_\varepsilon^1 (1 - \alpha\lambda)^{n/2}(1 - \beta\lambda)^{n/2}\ dE_\lambda x^*\right\| \le q^{n/2}(\varepsilon)\left\|\int_\varepsilon^1 dE_\lambda x^*\right\| \to 0, \ n \to \infty,$ since for $\lambda \in [\varepsilon, 1]$

$|(1 - \alpha\lambda)(1 - \beta\lambda)| \le q(\varepsilon) < 1.$ A $\left\|\int_0^\varepsilon (1 - \alpha\lambda)^{n/2}(1 - \beta\lambda)^{n/2}\ dE_\lambda x^*\right\| \le \left\|\int_0^\varepsilon dE_\lambda x^*\right\| = \left\|E_\varepsilon x^*\right\| \to 0, \varepsilon \to 0$

due to the properties of the spectral function.

From the condition (19) $n(\delta + \eta)^2 \to 0$ by $n \to \infty, \quad \delta \to 0, \quad \eta \to 0.$

From here $(\alpha + \beta)^{1/2} n^{1/2}\left(\delta + \eta\left\|x^*\right\|\right) \to 0, \ n \to \infty, \quad \delta \to 0, \quad \eta \to 0.$

Thus, $\left\|x_{n(\delta,\eta)} - x^*\right\| \to 0, \ n \to \infty, \quad \delta \to 0, \quad \eta \to 0.$

Theorem 3 has been proved.

Fair

Theorem 4. *Let* $\|A-A_\eta\|\le\eta$, $\|A_\eta\|^2\le 1$, $(0<\eta\le\eta_0)$, $y\in R(A)$, $\|y_\delta-y\|\le\delta$. *If the exact solution is represented as* $x^*=|A|^s z$, $s>0$, $\|z\|\le\rho$, $|A|=\left(A^*A\right)^{1/2}$ *and the conditions are met* $0<\alpha<2$, (6), (8), (9), (12), (13), *then the error estimate is fair*

$$\left\|x_{n(\delta,\eta)}-x^*\right\|\le\gamma_0 c_s\left(1+|\ln\eta|\right)\eta^{\min(1,s)}\rho+\gamma_{s/2}n^{-s/2}\rho+(\alpha+\beta)^{1/2}n^{1/2}\left(\delta+\left\|x^*\right\|\eta\right),\ 0<s<\infty.$$

<u>*Proof.*</u>

In the case of a source-like representable exact solution $x^*=|A|^s z=\left(A^*A\right)^{s/2}z$ from (13) we will get $\sup\limits_{0\le\lambda\le1}\lambda^{s/2}|1-\lambda g_n(\lambda)|\le\gamma_{s/2}n^{-s/2}$, where $\gamma_{s/2}=\left(\dfrac{s}{2(\alpha+\beta)}\right)^{s/2}$.

Then

$$\left\|K_{n\eta}\left|A_\eta\right|^s z\right\|=\left\|\left|A_\eta\right|^s\left[I-A_\eta^*A_\eta g_n\left(A_\eta^*A_\eta\right)\right]z\right\|=\left\|\left(A_\eta^*A_\eta\right)^{s/2}\left[I-A_\eta^*A_\eta g_n\left(A_\eta^*A_\eta\right)\right]z\right\|\le\gamma_{s/2}n^{-s/2}\rho.$$

From here

$$\left\|K_{n\eta}x^*\right\|=\left\|K_{n\eta}|A|^s z\right\|=\left\|K_{n\eta}\left(\left|A_\eta\right|^s-|A|^s\right)z\right\|+\left\|K_{n\eta}\left|A_\eta\right|^s z\right\|\le\gamma_0 c_s\left(1+|\ln\eta|\right)\eta^{\min(1,s)}\rho+\gamma_{s/2}n^{-s/2}\rho,$$

since from [3, p. 92] we have $\left|\left|A_\eta\right|^s-|A|^s\right|\le c_s\left(1+|\ln\eta|\right)\eta^{\min(1,s)}$, $c_s=\text{const}$ ($c_s\le2$ by $0<s\le1$). From (20) we have

$$\left\|x_{n(\delta,\eta)}-x^*\right\|\le\left\|K_{n\eta}x^*\right\|+\gamma_*n^{1/2}\left(\delta+\left\|x^*\right\|\eta\right)=\left\|K_{n\eta}x^*\right\|+(\alpha+\beta)^{1/2}n^{1/2}\left(\delta+\left\|x^*\right\|\eta\right)\le$$
$$\le\gamma_0 c_s\left(1+|\ln\eta|\right)\eta^{\min(1,s)}\rho+\gamma_{s/2}n^{-s/2}\rho+(\alpha+\beta)^{1/2}n^{1/2}\left(\delta+\left\|x^*\right\|\eta\right),\ 0<s<\infty.\tag{21}$$

Theorem 4 has been proved.

Minimizing the right-hand side (21) by *n*, we obtain the value of the a priori stopping moment:

$$n_{\text{opt}}=\left(\frac{s\gamma_{s/2}}{\gamma_*}\right)^{2/(s+1)}\rho^{2/(s+1)}\left(\delta+\left\|x^*\right\|\eta\right)^{-2/(s+1)}=$$
$$=2^{-s/(s+1)}s^{(s+2)/(s+1)}(\alpha+\beta)^{-1}\rho^{2/(s+1)}\left(\delta+\left\|x^*\right\|\eta\right)^{-2/(s+1)}.$$

Substituting n_{opt} into the estimate (21), we obtain the optimal error estimate for the iteration method (15):

$$\left\|x_{n(\delta,\eta)}-x^*\right\|_{\text{opt}}\le\gamma_0 c_s\left(1+|\ln\eta|\right)\eta^{\min(1,s)}\rho+c_s''\rho^{1/(s+1)}\left(\delta+\left\|x^*\right\|\eta\right)^{s/(s+1)},\ 0<s<\infty,$$

where $c_s''=\left(s^{1/(s+1)}+s^{-s/(s+1)}\right)\gamma_*^{s/(s+1)}\gamma_{s/2}^{1/(s+1)}=(2s)^{-s/(2(s+1))}(s+1)$.

Thus,

$$\left\|x_{n(\delta,\eta)}-x^*\right\|_{\text{opt}} \le c_s\left(1+|\ln\eta|\right)\eta^{\min(1,s)}\rho+\left(2s\right)^{-s/(2(s+1))}(s+1)\rho^{1/(s+1)}\left(\delta+\left\|x^*\right\|\eta\right)^{s/(s+1)},\ 0<s<\infty.$$

Remark 3. *The optimal error estimate does not depend on* α *and* β*, but* $n_{опт}$ *depends on* α *and* β*. Therefore, in order to reduce the amount of computational work, it is necessary to take* α *and* β *as much of the conditions as possible* $0<\alpha<2$, (6), (8), (9), (11), (12), (13) *and in such a way that* $n_{опт}\in Z$.

Conclusion

In this article, some properties of the proposed explicit iteration scheme for solving ill-posed problems are studied: the convergence of approximations with an a priori choice of the regularization parameter in the initial norm of Hilbert space in the case of a bounded self-adjoint and non-self-adjoint inaccurately specified operator is proved, error estimates and estimates for stopping moments are obtained.

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